Ultrametrics Meet Fine-Grained Complexity

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Joint work with

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(Google)

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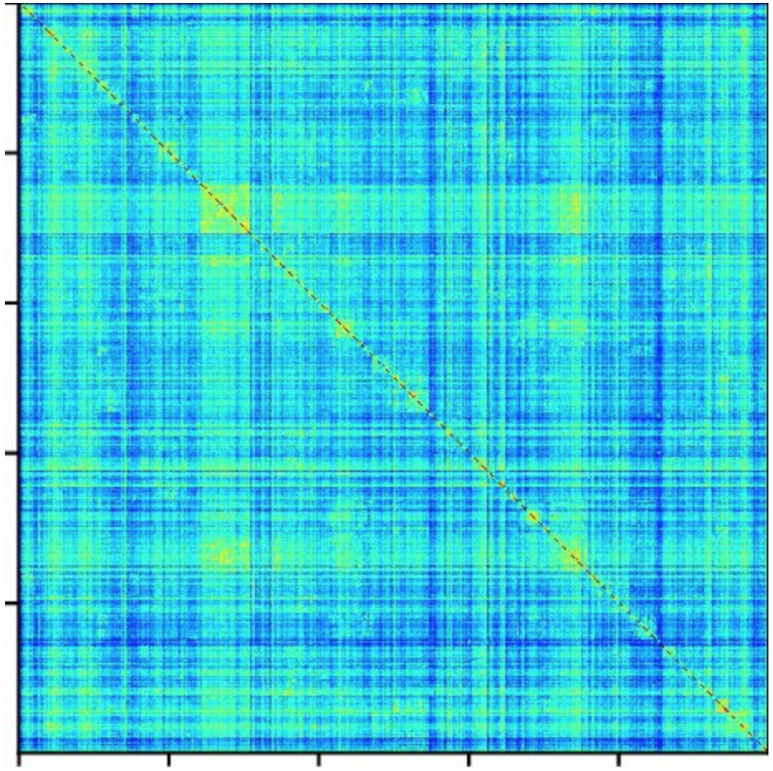
(LaBRI)

How to Visualize all this data?



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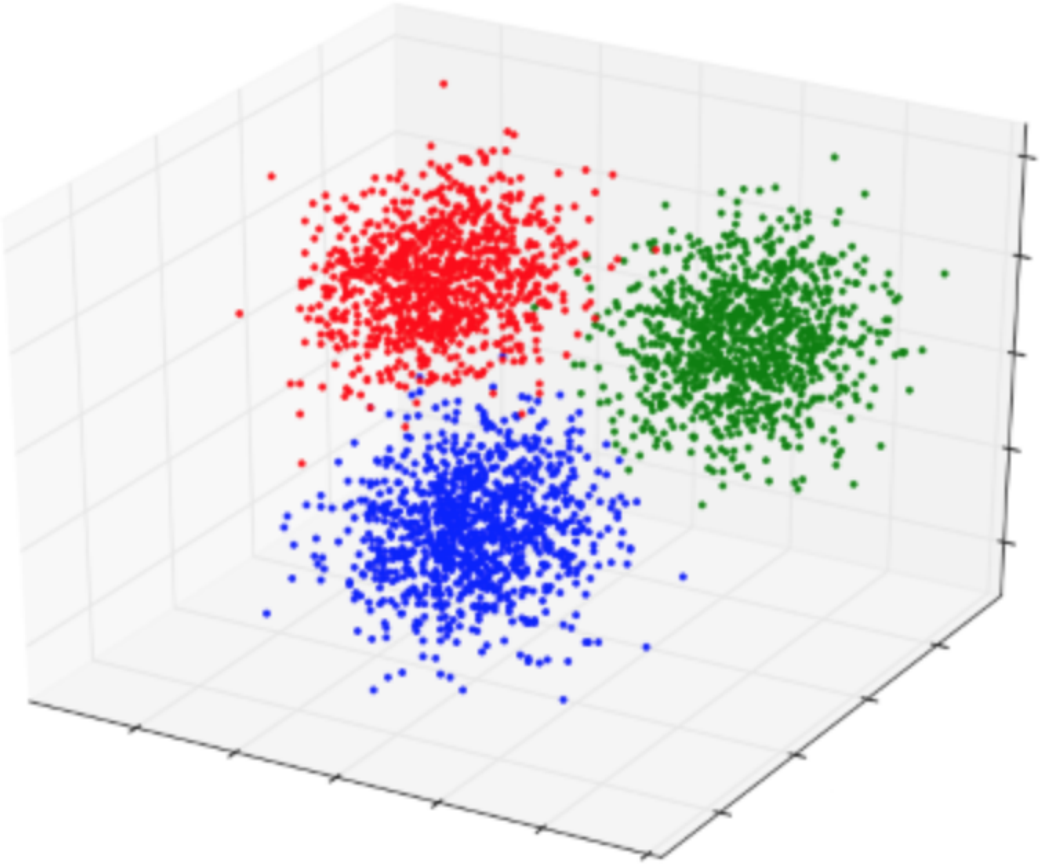

Approaches

- Principal Component Analysis

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- Dendrogram / Hierarchical Clustering

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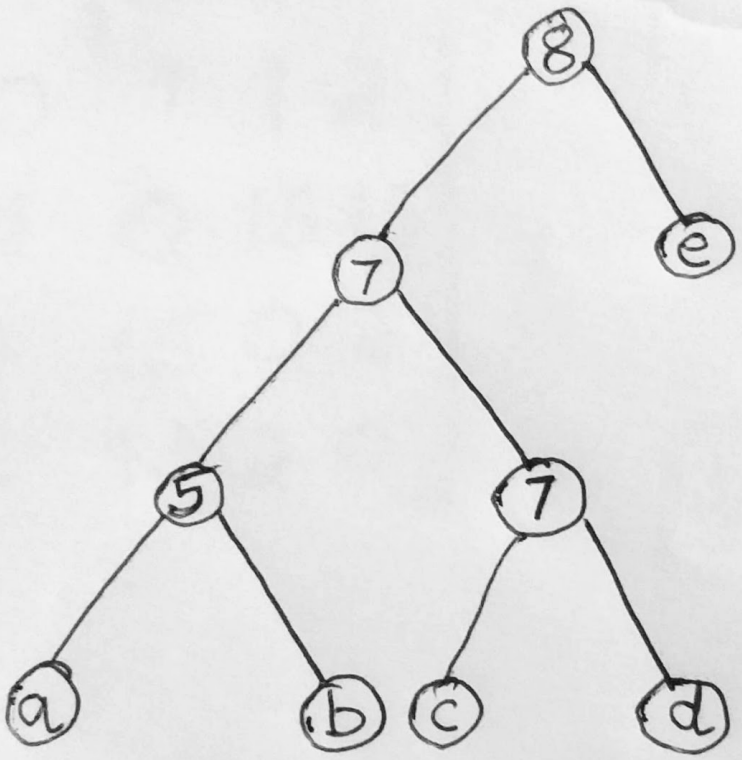
- Cool Property: $\forall a, b, c \in \Gamma$,

$$\Delta(a,b) = \Delta(a,c) \text{ or } \Delta(a,c) = \Delta(b,c) \text{ or } \Delta(a,b) = \Delta(b,c)$$

Example



Δ: Leaves $\to \mathbb{R}^+$

$\Delta(x,y) = w(LCA(x,y))$

w is non-increasing from root

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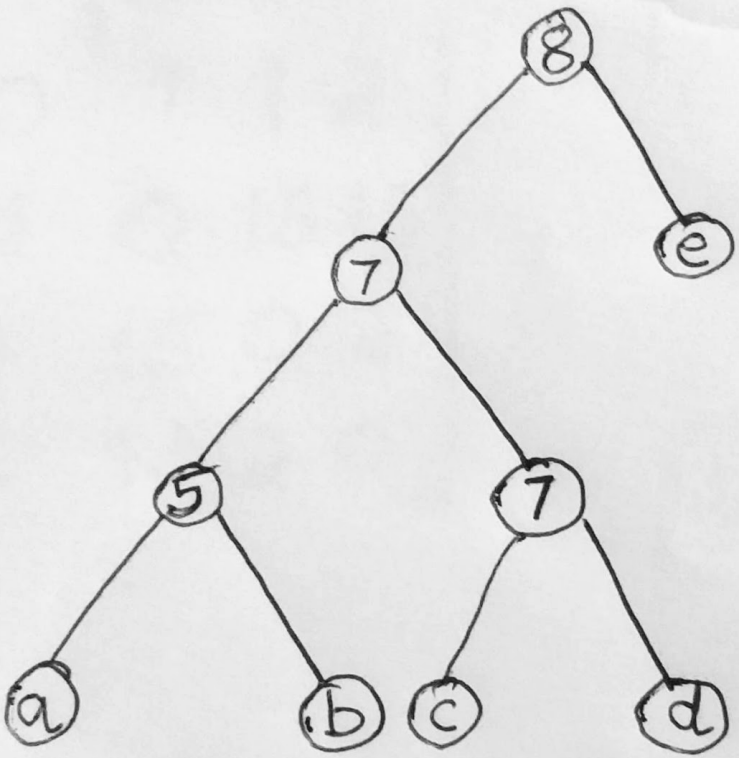

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Arises in:

- Evolutionary Biology
- Hierarchical Clustering

Example

- Topology: Discrete metric
- Number Theory: p-adic numbers
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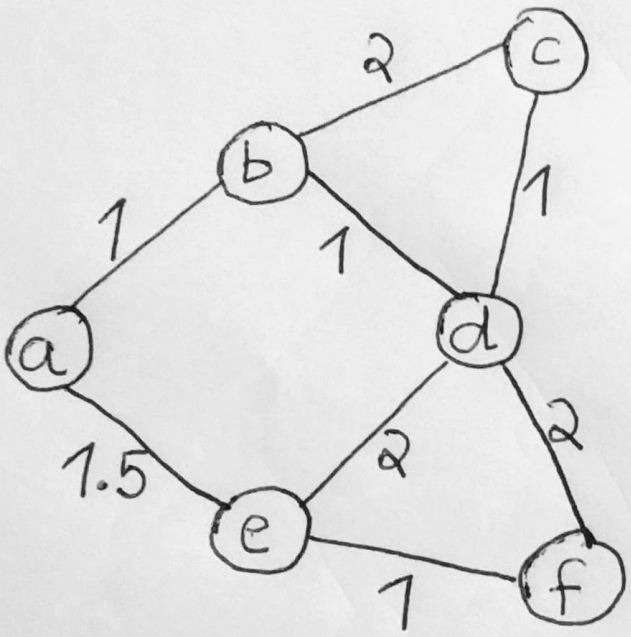

$$\Delta : V \to \mathbb{R}^+$$

$$\Delta(x,y) = \min_{\substack{\text{paths } p \\ x \leftrightarrow y}} \max_{e \in p} w(e)$$

$$\Delta(c,e) = 1.5$$

Directions

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$$\tau : X \to L, \ \ \forall x, y \in X,$$

$$\|x - y\|_p \leq \Delta(\tau(x), \tau(y)) = w(\mathsf{LCA}(\tau(x), \tau(y))) \leq \rho_{\mathsf{OPT}} \cdot \|x - y\|_p$$

Motivation: Data Visualization

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$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 1.5 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1.5 & 1 & 0 & 1.5 & 1.5 \\ 2 & 1 & 1.5 & 0 & 2 \\ 1 & 2 & 1.5 & 2 & 0 \end{bmatrix} \end{array}$$

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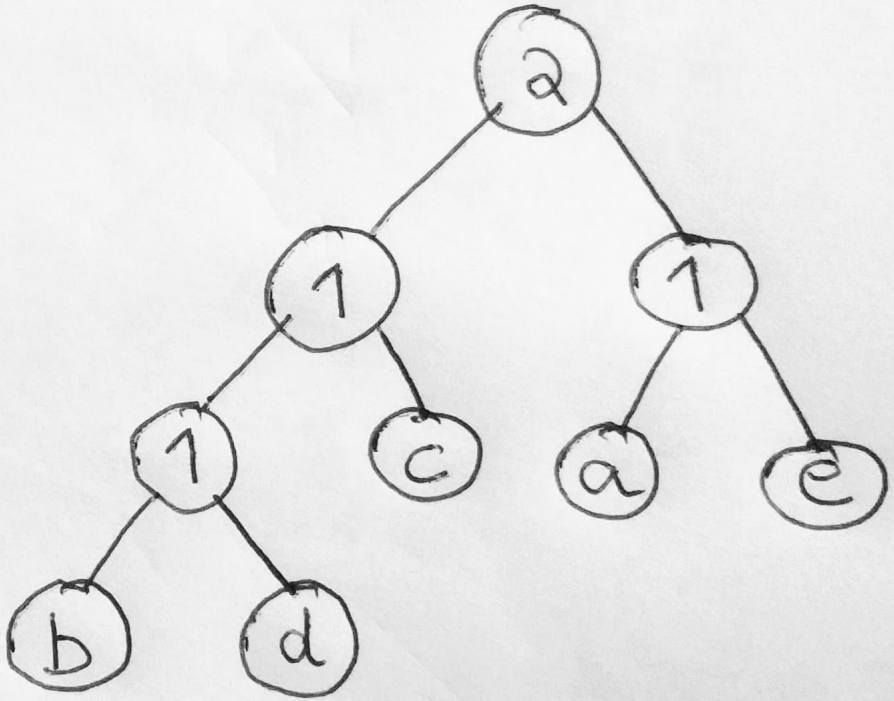
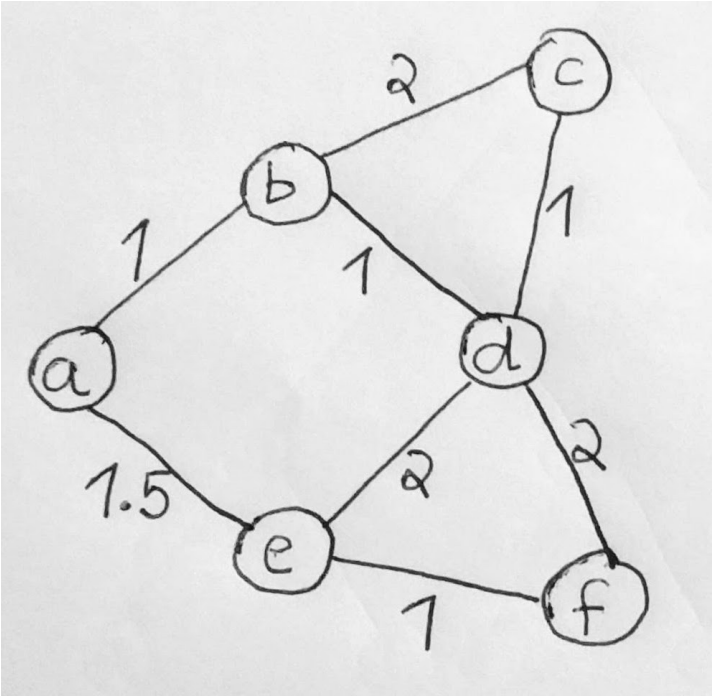


Intuition



Replace edge weights with min-max path weights

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Performs Well in Experiments!

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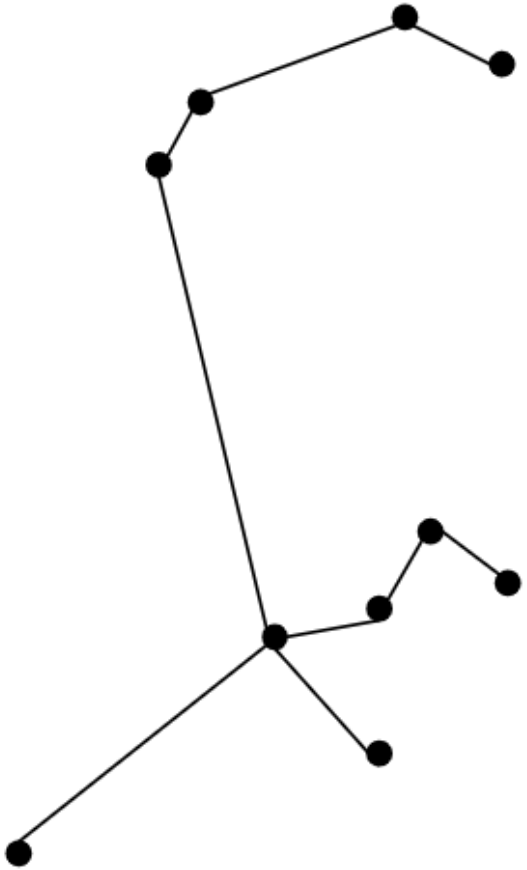

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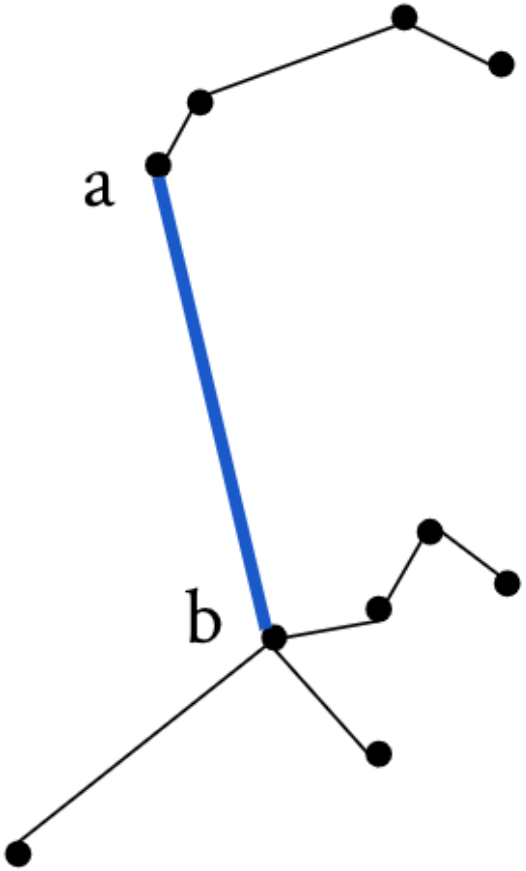

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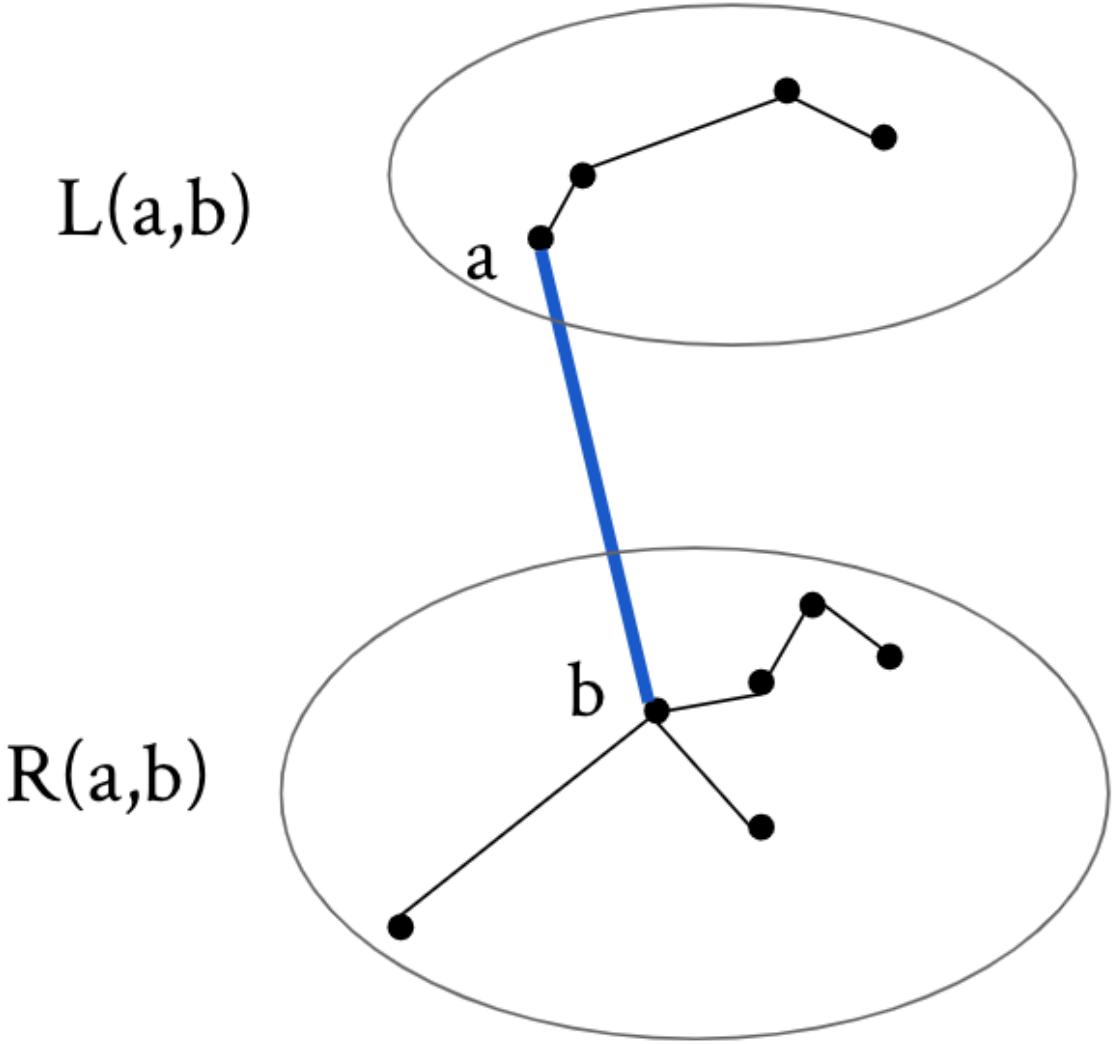

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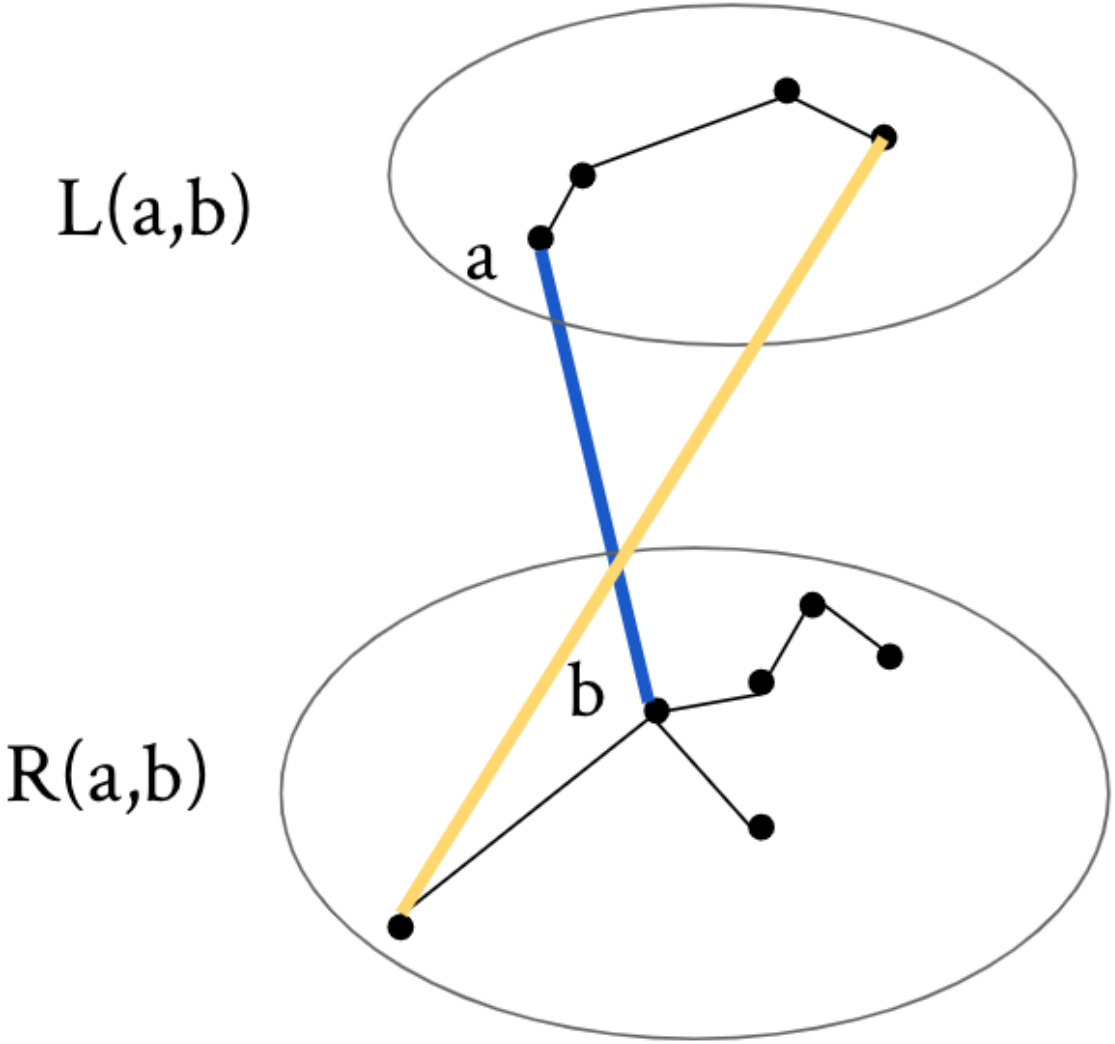

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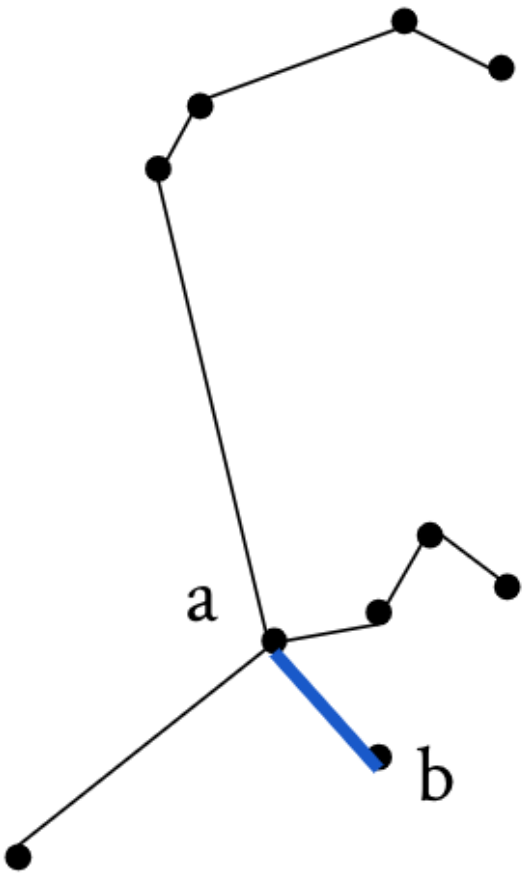

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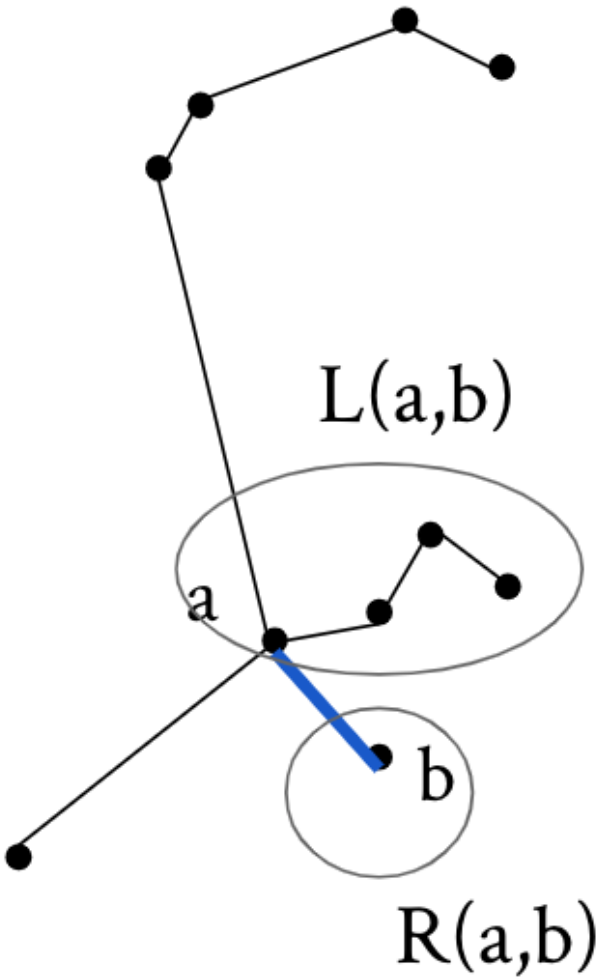

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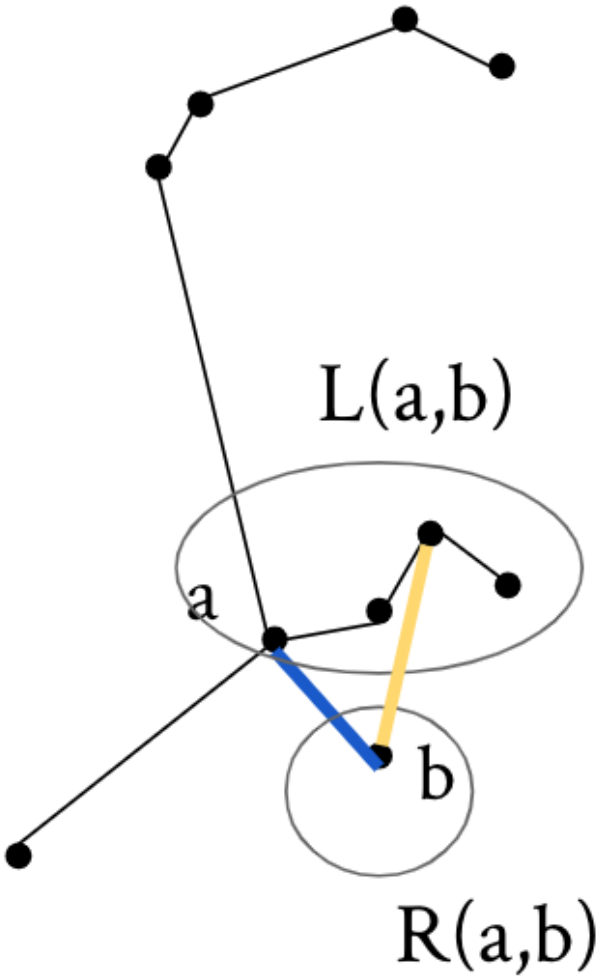

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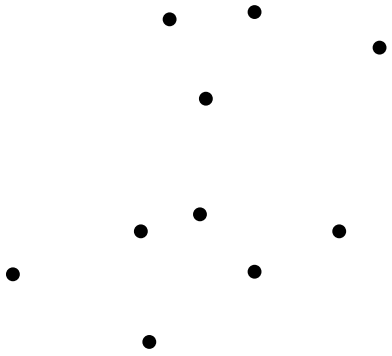

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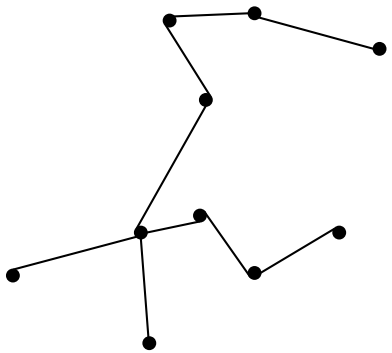

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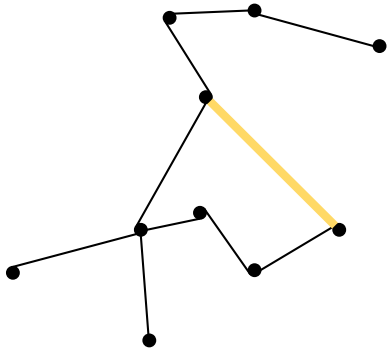

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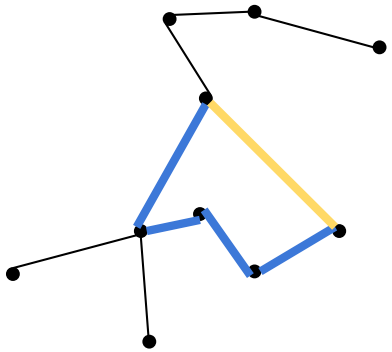

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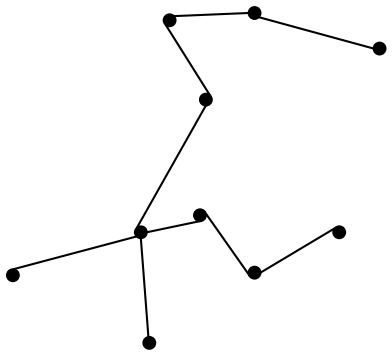

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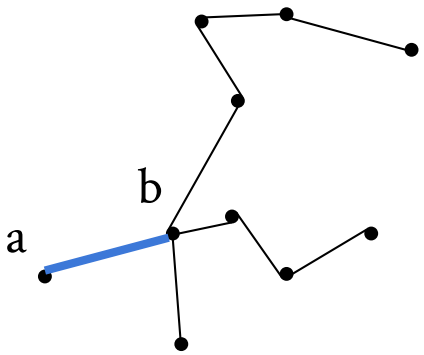

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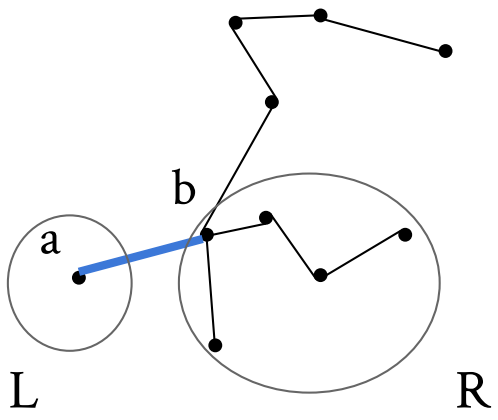

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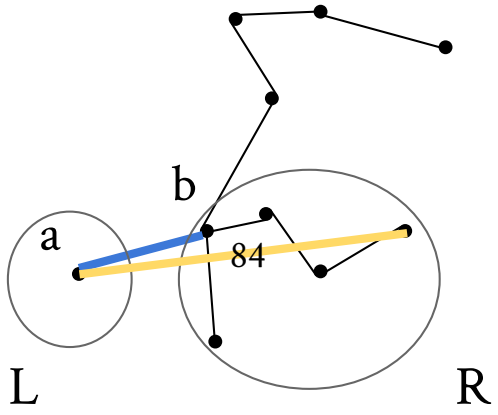

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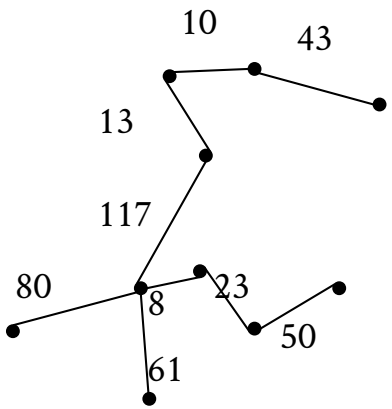

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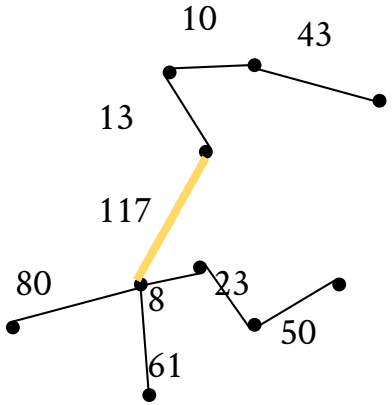

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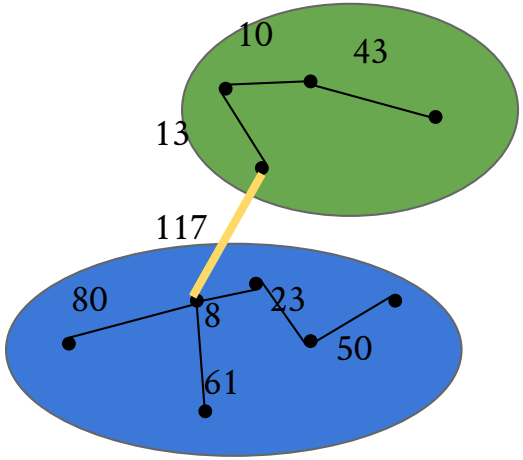

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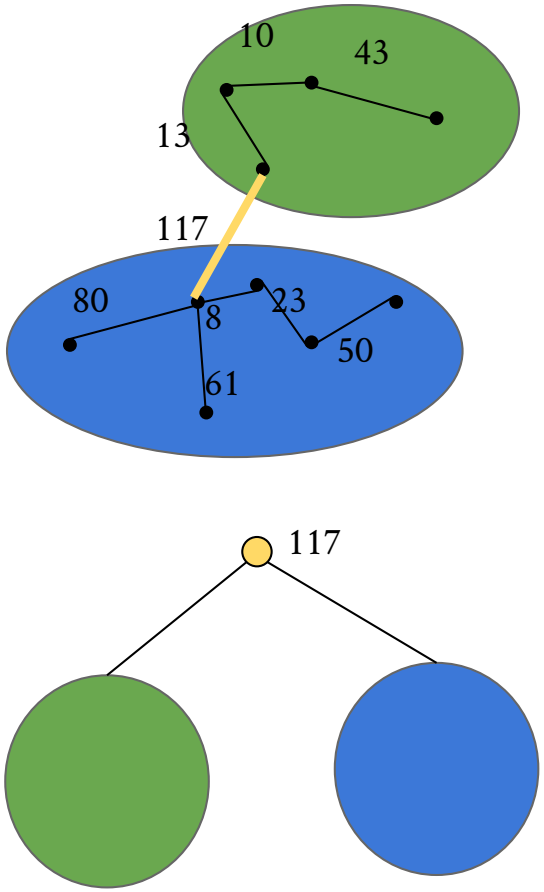

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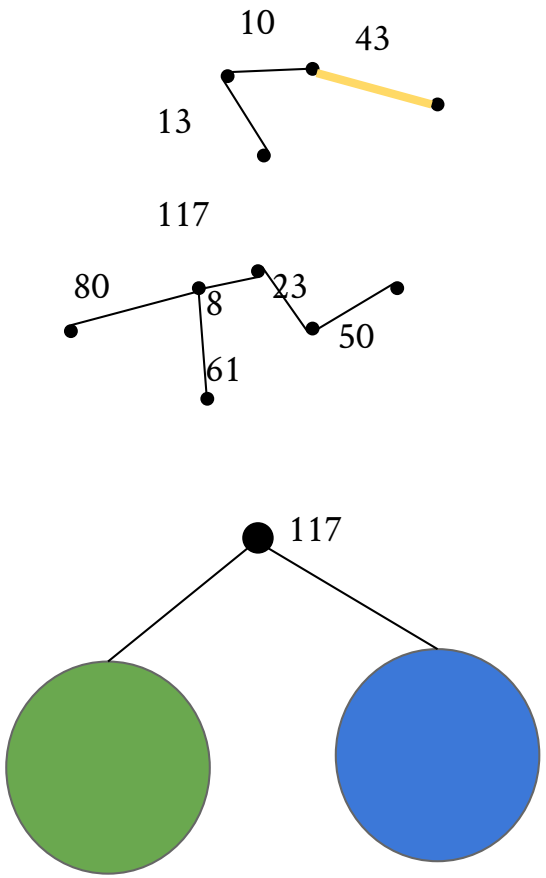

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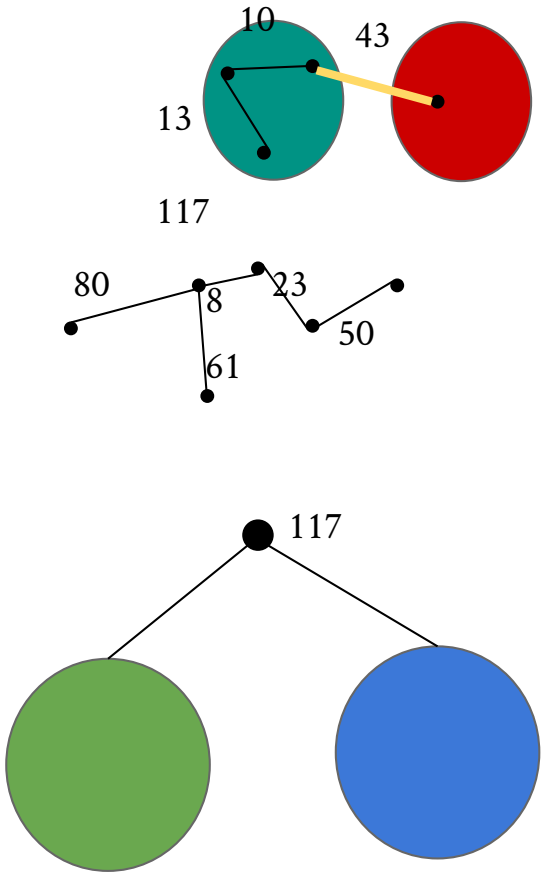

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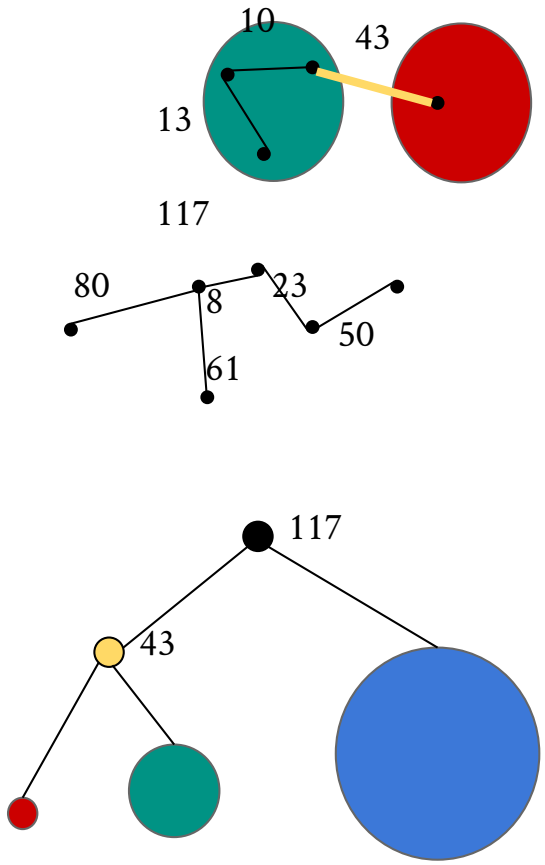

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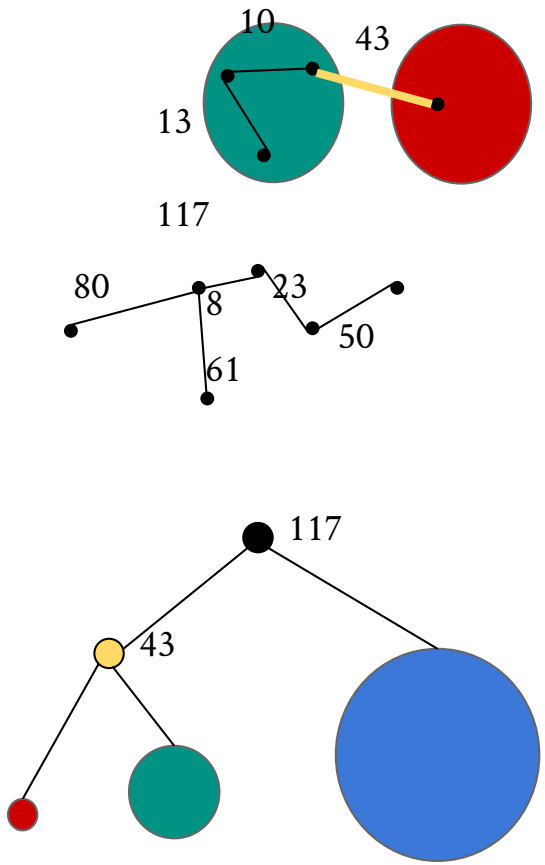

1. Compute a γ-approximate MST T^G over the complete graph G
2. Compute a β-estimate of the cut weights of the edges in T^G
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→ This gives a $\gamma \cdot \beta$-approximation

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$$\forall v_i, v_j \in V,\ \Delta'(v_i, v_j) \leq \Delta(v_i, v_j) \leq \beta \cdot \Delta'(v_i, v_j)$$

Our Approximation Algorithm: Implementation

- For any $\gamma \geq 1$, γ-spanner constructions of Har-Peled, Indyk, Sidiropoulos'13 in time $O(nd + n^{1+O(1/\gamma^2)})$

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Implementation: Approximate Euclidean MST

- A γ-spanner of $S \in \mathbb{R}^d$ is $G(S, E)$:

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- Kruskal Algorithm on G gives γ-approximate MST T of G

$$\forall e \in G \setminus T,\ w(e) \geq \frac{1}{\gamma} \cdot \max_{e' \in C_e^T} w(e').$$

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- For $e = (x, y) \in C \times D$

$$5 \cdot \max(d(r_C, r_D), m_C - d(r_C, r_D), m_D - d(r_C, r_D))$$

is a 5 approximation of $CW(e)$.

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  - ApproxULT (approximate MST)
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	DIABETES	MICE	PENDIGITS
Average	11.1	9.7	27.5
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- Single linkage is almost optimal
- Room for improvement: Good approximate MST

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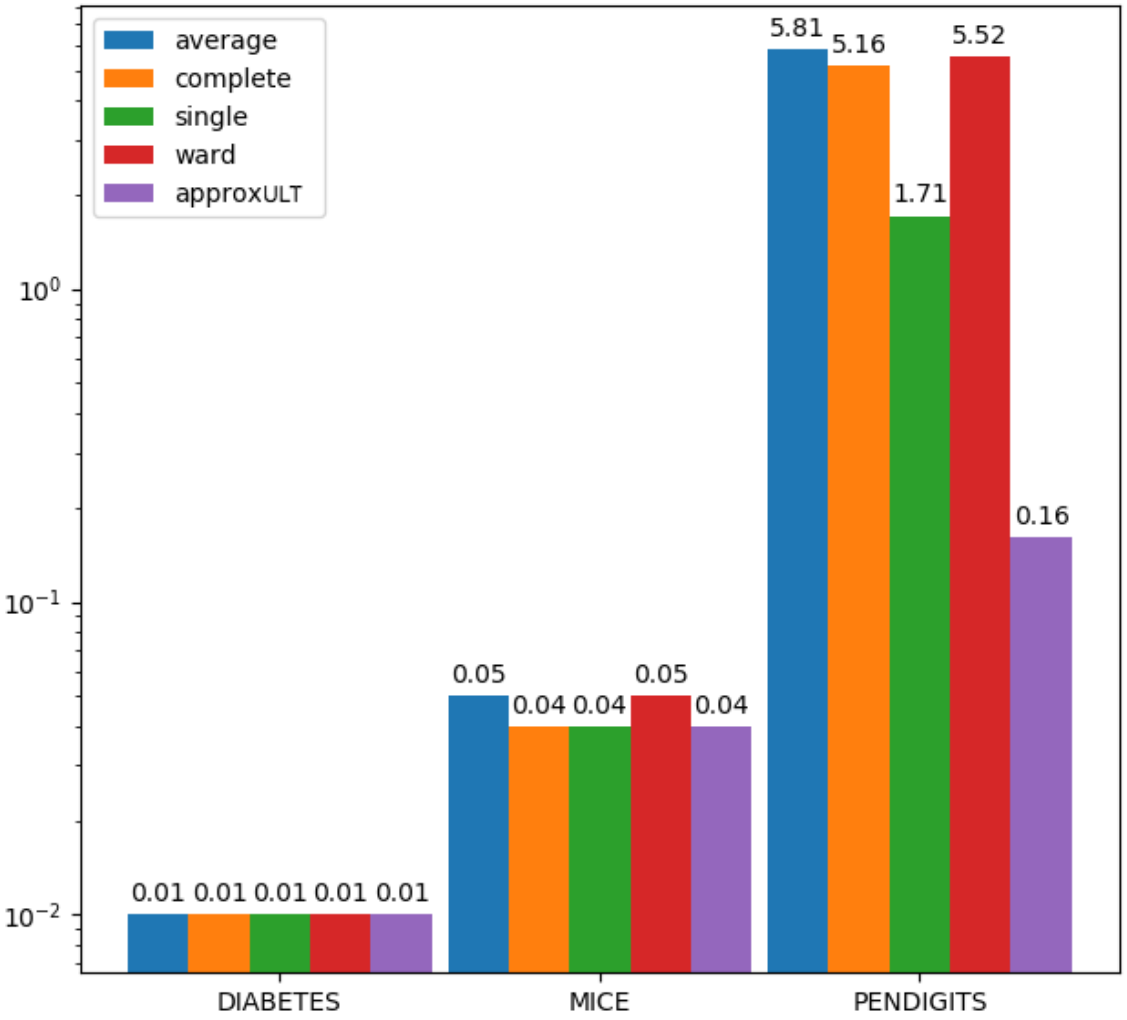

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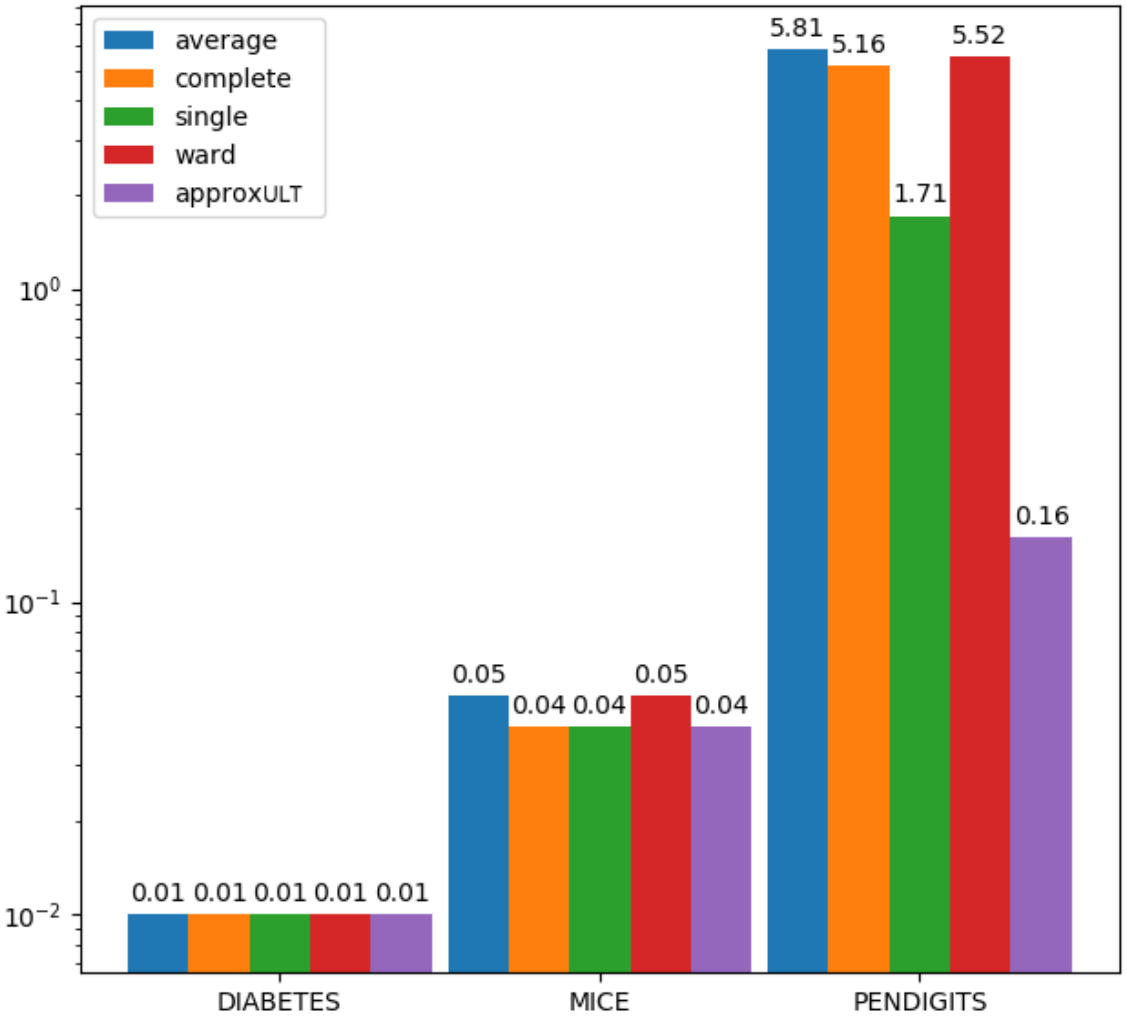

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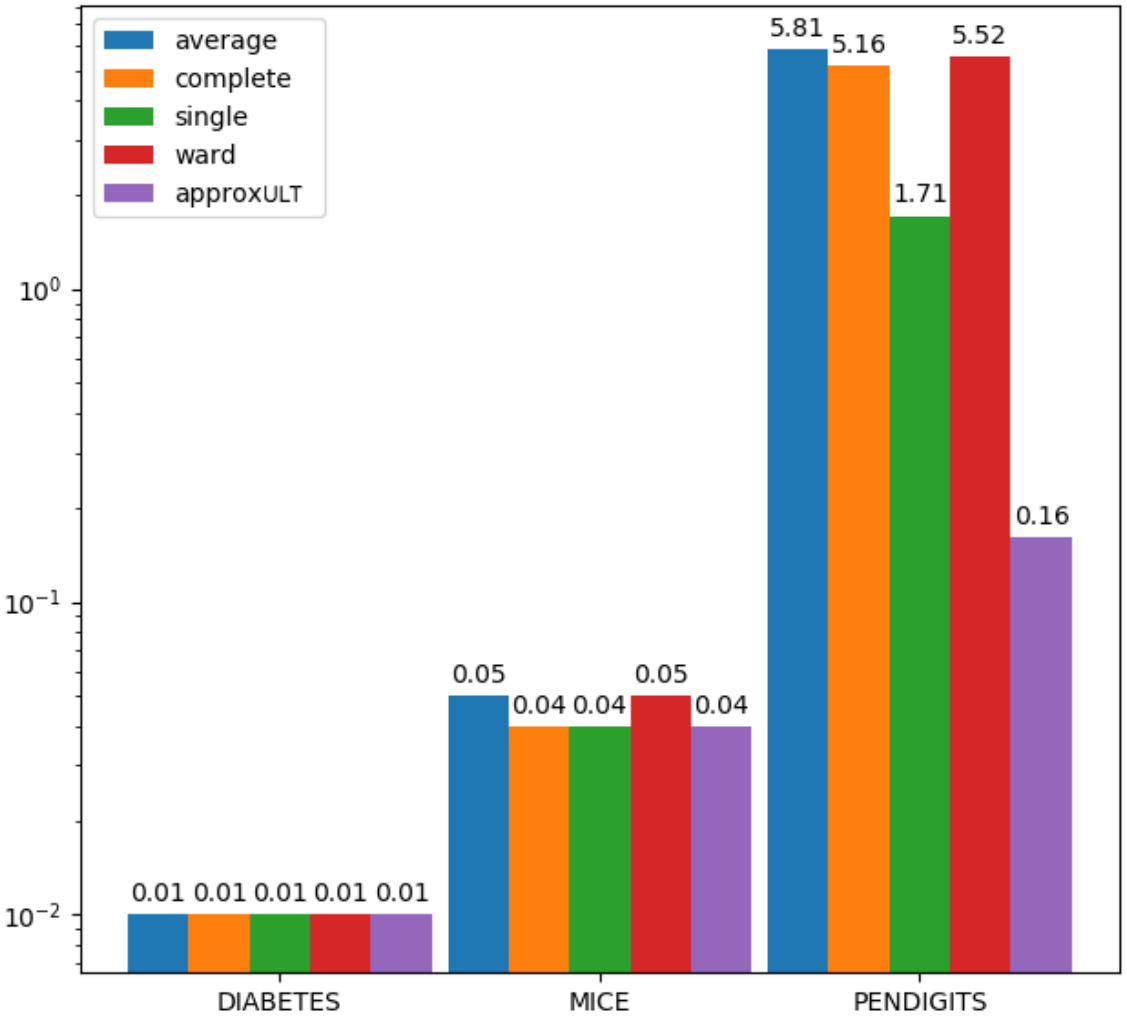

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- `https://github.com/guillaume-lagarde/fast-ultrametrics`
  - Helped also by Rémi de Verclos

Hardness from SETH

Theorem (Cohen-Addad–K–Lagarde)

Assuming SETH, for every $\varepsilon > 0$, no algorithm running in $n^{2-\varepsilon}$ time, given $X \in \mathbb{R}^{O_\varepsilon(\log n)}$ ($|X| = n$) in ℓ_∞-space can distinguish:

YES: X can be embedded isometrically into an ultrametric.
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Morally Equivalent to Search Version

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$$\Uparrow$$

Theorem (David–K–Laekhanukit'19)

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YES: $\exists (a, b) \in A \times B$ such that $\|a - b\|_\infty = 1$.
NO: $\forall (a, b) \in A \times B$ we have $\|a - b\|_\infty = 3$.

Moreover, in both cases $\mathrm{dist}(A) = \mathrm{dist}(B) = 2$ and $\mathrm{dist}(A, B) \in \{1, 3\}$.

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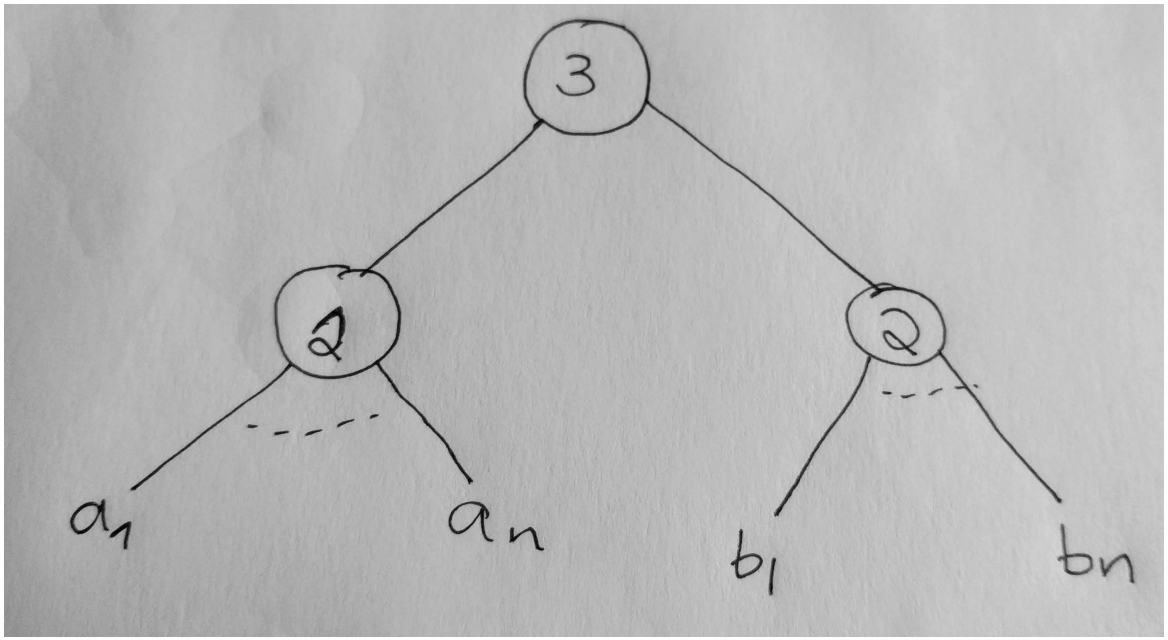

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$$\begin{aligned} 3 = \|a' - b\|_\infty &\leq \Delta(\tau(a'), \tau(b)) \\ &\leq \max\{\Delta(\tau(a), \tau(b)), \Delta(\tau(a'), \tau(a))\} \\ &\leq \max\{\rho \cdot \|a - b\|_\infty, \rho \cdot \|a' - a\|_\infty\} = 2\rho \end{aligned}$$

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Results

Theorem (Farach–Kannan–Warnow'95)

Given the distance matrix of n points, the optimal ultrametric embedding can be computed in time $O(n^2)$.

Theorem (Cohen-Addad–K–Lagarde)

- Assuming SETH, no 1.5 approximate embedding in $n^{1.99}$ time from ℓ_∞-metric.
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Performs Well in Experiments!

Open Problems

Improved Approximation Factor?

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Euclidean Inapproximability under SETH?

THANK YOU!