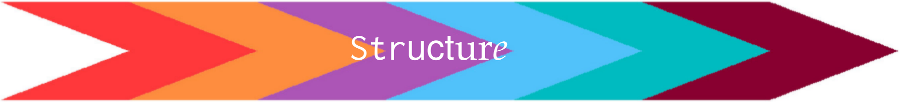
Hardness of Approximation for Metric Clustering

Karthik C. S.

(New York University)

June 22^{nd} 2021

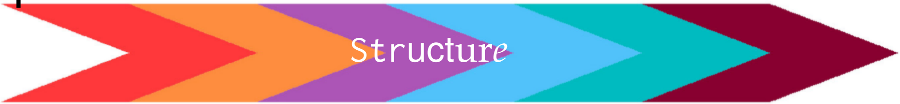
Spectrum of Computational Problems



Spectrum of Computational Problems

Circuit-SAT

CSP



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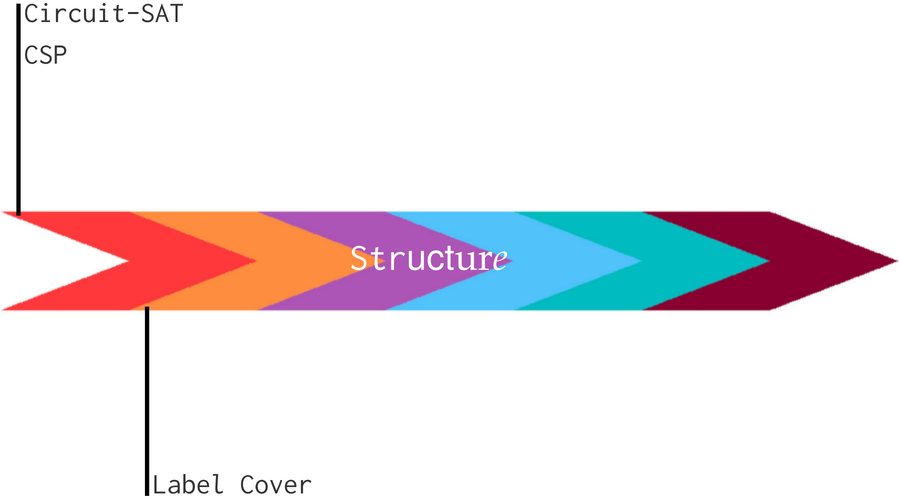

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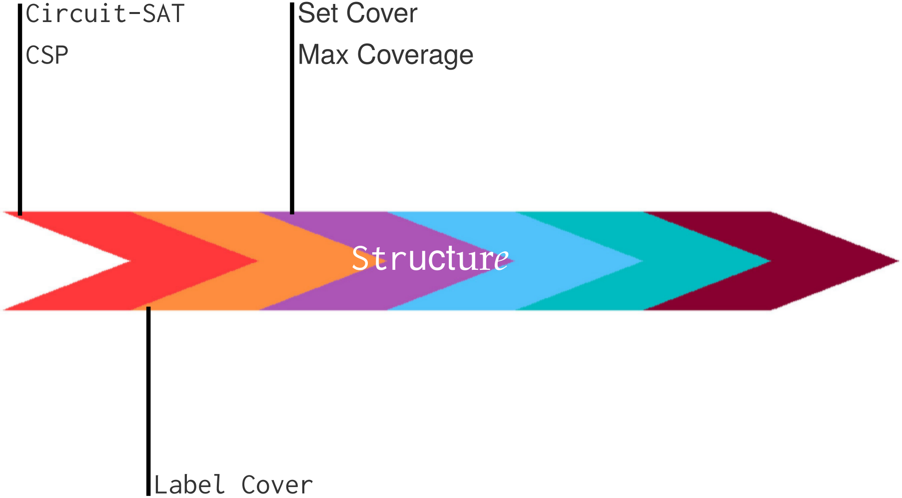

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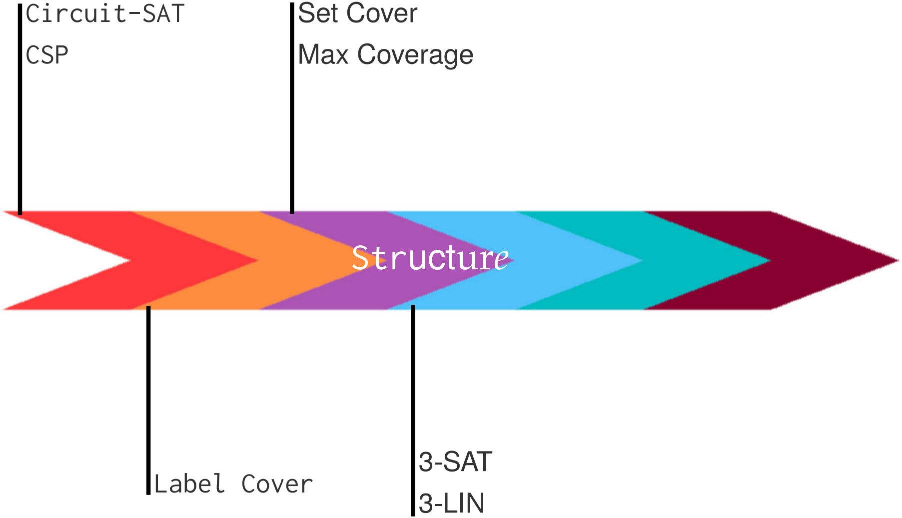

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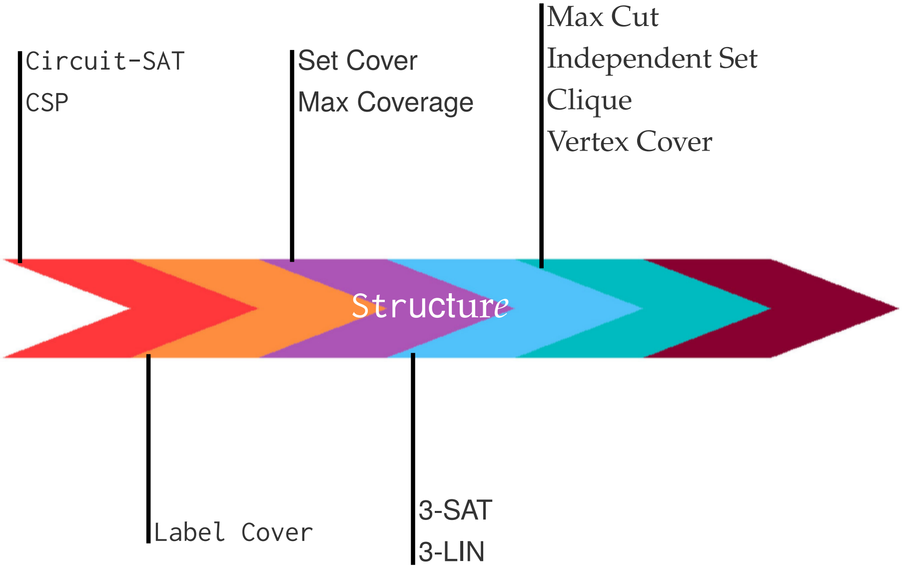

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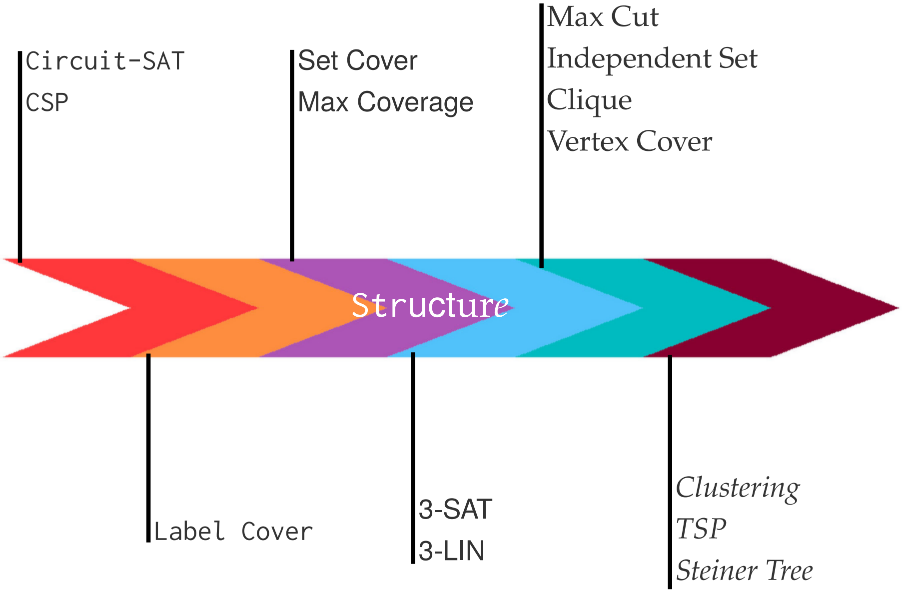

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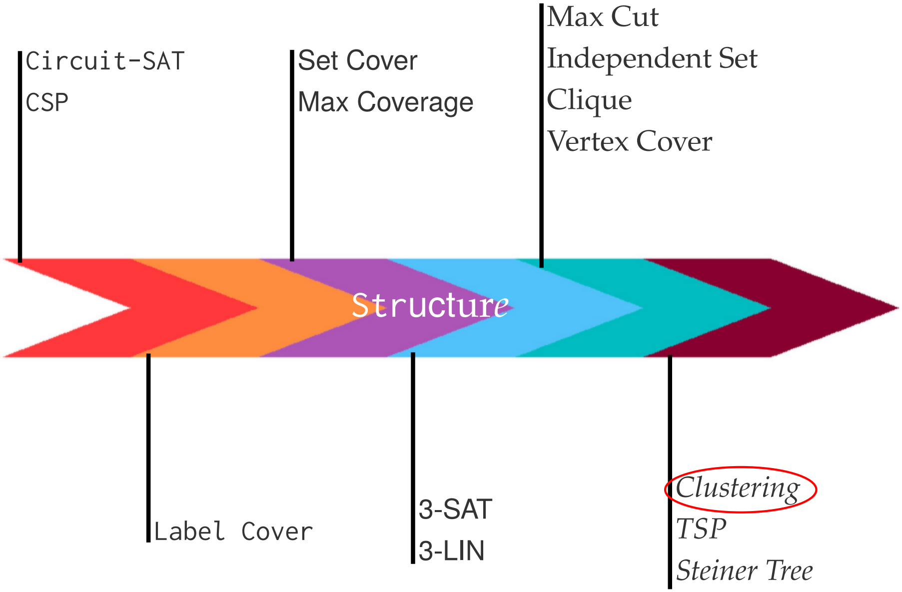

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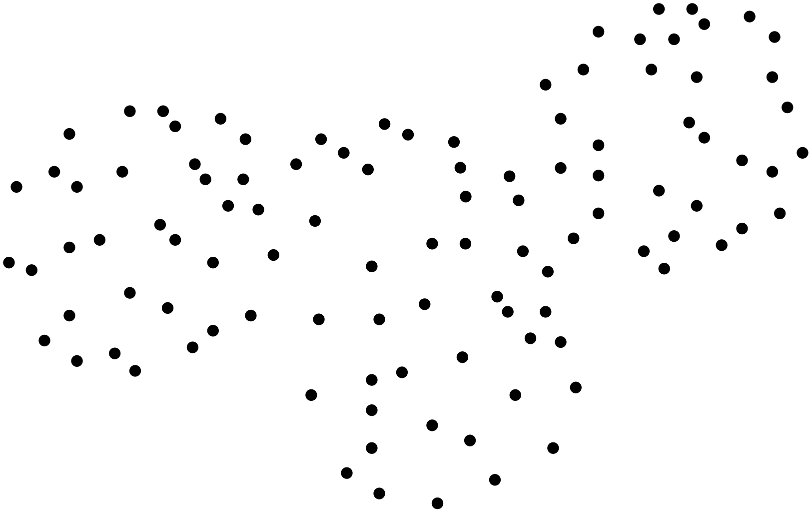

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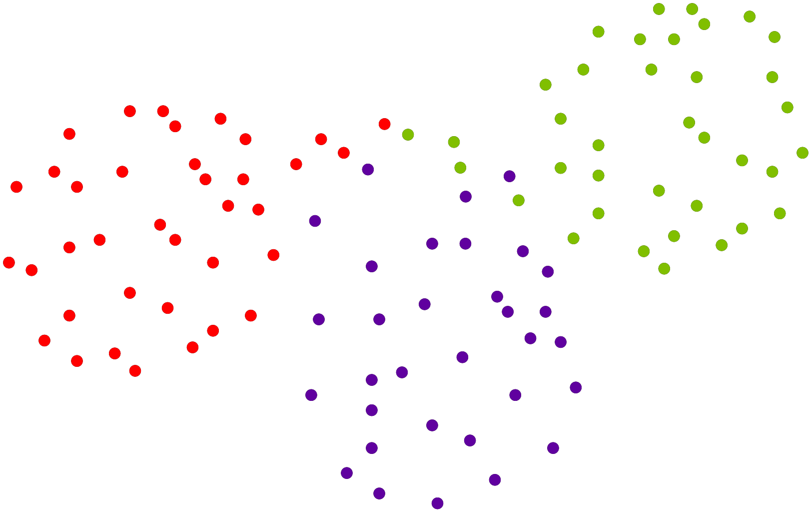

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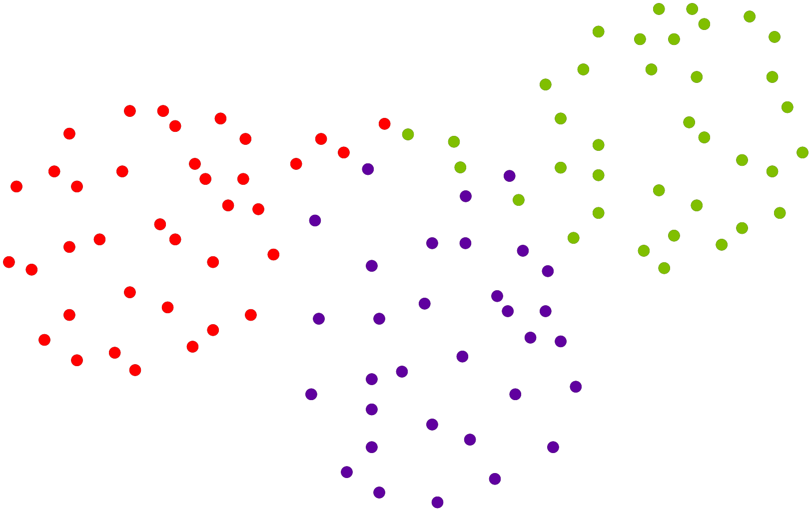

Task of Classifying Input Data

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- **General metric**: k-means $\geq$ 9
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Approximation Algorithms

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  - Fixed k: PTAS (Kumar–Sabharwal–Sen'10)

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Hardness of Approximation: Before 2019

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- ℓ_2-metric: k-means < 1.0013 (Lee-Schmidt-Wright'17)

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Continuous is Computationally Easier than Discrete?

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State-of-the-art for k-means

Discrete Version

	JCH	UGC	NP≠P
ℓ_1-metric	3.94	1.56	1.38
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Continuous Version

General metric ≈ 4 (NP≠P)

ℓ_2-metric ≈ 1.36 (JCH), 1.07 (UGC), 1.06 (NP≠P)

ℓ_1-metric ≈ 2.10 (JCH), 1.16 (NP≠P)

Warm Up: General Metrics

Theorem (Guha-Khuller'99)

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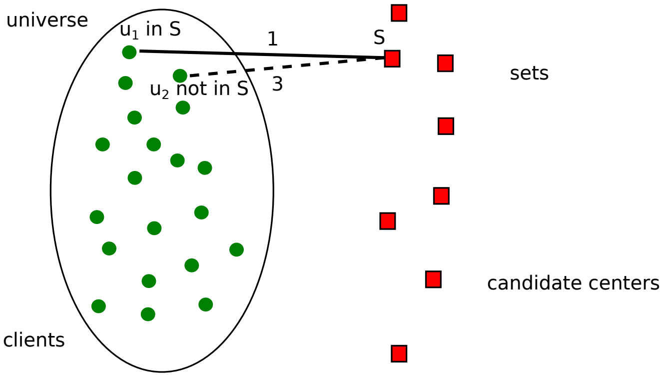

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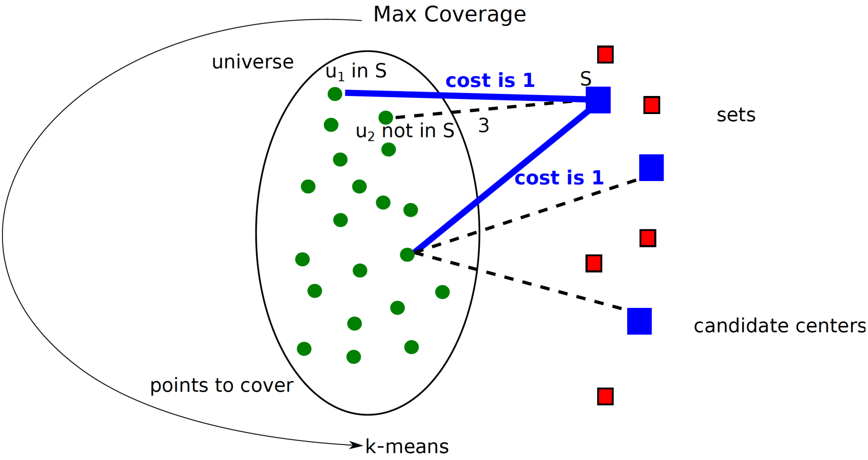

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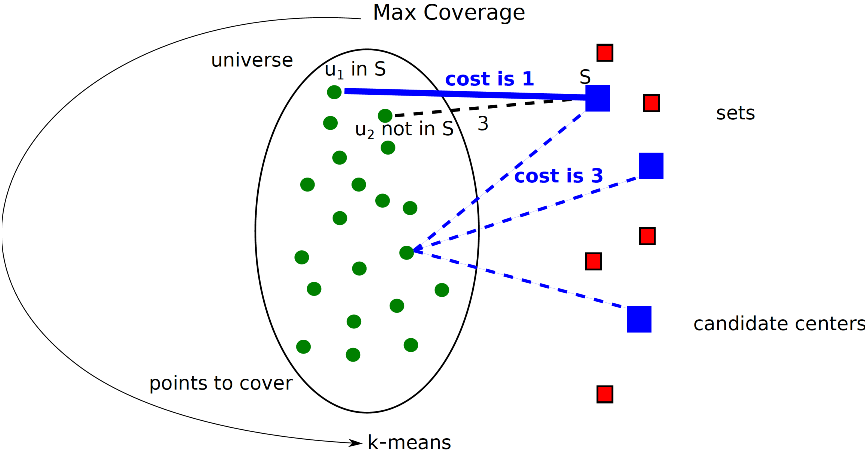

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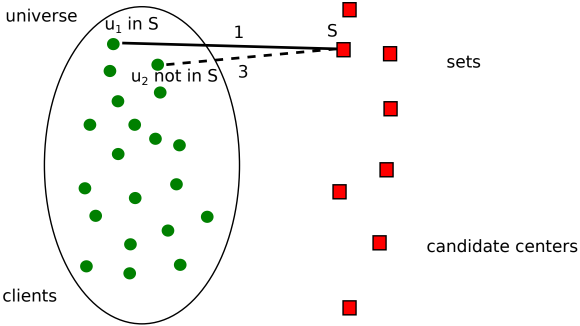
Johnson Coverage Hypothesis



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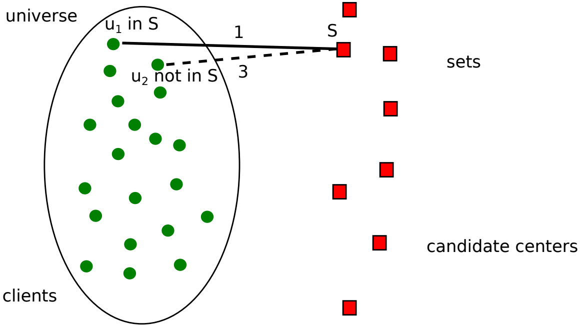

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even when set system is induced subgraph of **Johnson graph**.

Johnson Coverage Hypothesis

(α, t)-Johnson Coverage Problem

Given $E \subseteq \binom{[n]}{t}$, and k as input, distinguish between:

Completeness: There exists $\mathscr{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ such that

$$\forall T \in E,\ \exists S_i \in \mathscr{C},\ S_i \subset T.$$

Soundness: For every $\mathscr{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ we have

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Johnson Coverage Hypothesis (Cohen-Addad–K–Lee)

$\forall \varepsilon > 0, \exists t_\varepsilon \in \mathbb{N}$ such that $(1 - \frac{1}{e} + \varepsilon, t_\varepsilon)$-Johnson Coverage problem is NP-hard.

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- 3-Hypergraph Vertex Coverage problem is NP-Hard to approximate to a factor of $7/8$

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- Johnson Graph Embedding into ℓ_p-metrics

Embedding in Hamming metric

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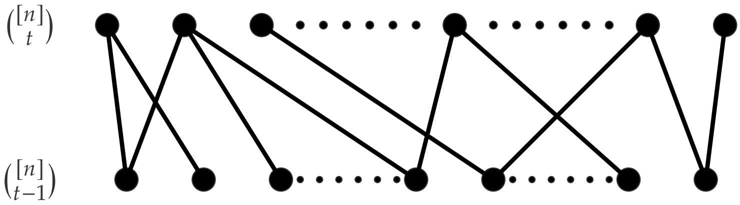
Johnson Graph Embedding



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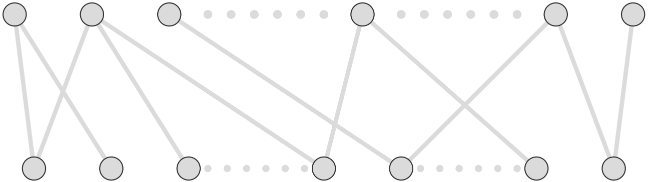

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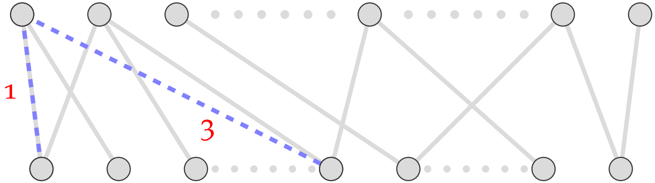

Proof Framework

3 ingredients

- JCH instance
- Dimensionality reduction for all ℓ_p-metrics
  - Works only for JCH instances
  - Arises from transcript of a communication game
- Johnson Graph Embedding into ℓ_p-metrics

State-of-the-art for k-means

Discrete Version

	JCH	UGC	NP≠P
ℓ_1-metric	3.94	1.56	1.38
ℓ_2-metric	1.73	1.17	1.17
ℓ_∞-metric	3.94	3.94	3.94

Continuous Version

General metric ≈ 4 (NP≠P)

ℓ_2-metric ≈ 1.36 (JCH), 1.07 (UGC), 1.06 (NP≠P)

ℓ_1-metric ≈ 2.10 (JCH), 1.16 (NP≠P)

State-of-the-art for k-median

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ℓ_1-metric	1.73	1.14	1.12
ℓ_2-metric	1.27	1.07	1.07
ℓ_∞-metric	1.73	1.73	1.73

State-of-the-art for k-median

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General metric ≈ 2 (NP≠P)

ℓ_2-metric ≈ 1.08 (JCH*), 1.015 (NP≠P)

ℓ_1-metric ≈ 1.36 (JCH*), 1.07 (UGC), 1.06 (NP≠P)

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Key Ingredient: Hard Instances of Max-Coverage with large girth

Key Takeaways

Improved Inapproximability of

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THANK
YOU!

OPEN PROBLEMS

Johnson Coverage Hypothesis

(α, t)-Johnson Coverage Problem

Given $E \subseteq \binom{[n]}{t}$, and k as input, distinguish between:

Completeness: There exists $\mathcal{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ such that

$$\forall T \in E,\ \exists S_i \in \mathcal{C},\ S_i \subset T.$$

Soundness: For every $\mathcal{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ we have

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Johnson Coverage Hypothesis (Cohen-Addad–K–Lee)

$\forall \varepsilon > 0, \exists t_\varepsilon \in \mathbb{N}$ such that $(1 - \frac{1}{e} + \varepsilon, t_\varepsilon)$-Johnson Coverage problem is NP-hard.

Johnson Coverage Hypothesis: Discussion

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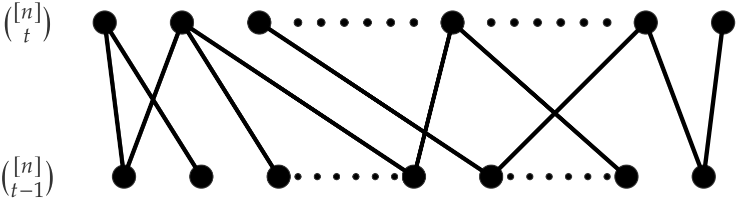
ℓ_∞-metric $\approx$???

Inapproximability of Clustering in Euclidean metrics



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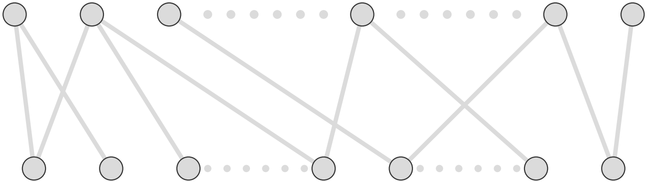

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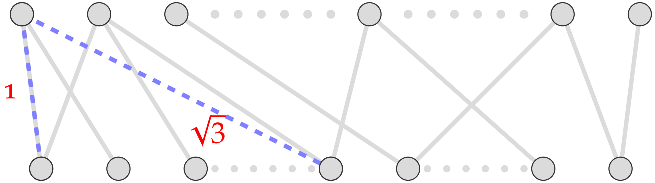

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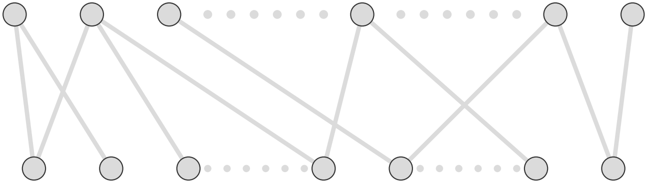

Is there a better embedding of the Johnson Graph into the Euclidean metric?

Other Open Problems

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