Reversing Color Coding

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Joint work with

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Colored vs. Uncolored Problems
Outline of Talk

- Colored vs. Uncolored Problems
- Closest Pair Problem
Outline of Talk

- Colored vs. Uncolored Problems
- Closest Pair Problem
- Parameterized Set Intersection Problem
Colored versus Uncolored
Uncolored $k$-Clique Problem:

**Input:** $G(V, E)$

**Output:** $k$-clique in $G$
Uncolored $k$-Clique Problem:

**Input:** $G(V,E)$

**Output:** $k$-clique in $G$

Colored $k$-Clique Problem:

**Input:** $G(V_1 \cup V_2 \cup \cdots \cup V_k, E)$

**Output:** $k$-clique in $G$ from $V_1 \times V_2 \times \cdots \times V_k$
Uncolored $k$-Clique Problem:

**Input:** $G(V, E)$

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**Input:** $G(V_1 \cup V_2 \cup \cdots \cup V_k, E)$

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Uncolored $k$-Clique Problem and Colored $k$-Clique Problem are computationally equivalent up to $O_k(1)$ factor
Uncolored $k$-Set Cover Problem:

**Input:** $S_1, \ldots, S_n \subseteq [n]$

**Output:** $S_{i_1}, \ldots, S_{i_k}$ whose union is $[n]$
Uncolored $k$-Set Cover Problem:

**Input:** $S_1, \ldots, S_n \subseteq [n]$

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Colored $k$-Set Cover Problem:

**Input:** $S^1_1, \ldots, S^1_n, S^2_1, \ldots, S^2_n, \ldots, S^k_1, \ldots, S^k_n \subseteq [n]$

**Output:** $S^1_{i_1}, \ldots, S^k_{i_k}$ whose **union** is $[n]$
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Uncolored and Colored $k$-Set Cover Problems are 
**computationally equivalent** up to $O_k(1)$ factor
Uncolored Clustering Problem:

Input: $P \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$

Output: $P_1 \cup P_2 \cup \cdots \cup P_k := P$ minimizing some clustering objective
Uncolored Clustering Problem:

**Input:** $P \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$

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Colored Clustering Problem:

**Input:** $P \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$, $c : P \rightarrow [r]$

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Input: \( P \subseteq \mathbb{R}^d, k \in \mathbb{N} \)

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Input: \( P \subseteq \mathbb{R}^d, k \in \mathbb{N}, c : P \rightarrow [r] \)

Output: \( P_1 \cup P_2 \cup \cdots \cup P_k := P \) minimizing some clustering objective such that each \( P_i \) is well-colored by \( c \)

Is Clustering under Fairness constraints computationally harder than Standard Clustering?
Uncolored Closest Pair Problem:

**Input:** $P \subseteq \mathbb{R}^d$

**Output:** $a, b \in P$ minimizing $\|a - b\|_p$
Uncolored Closest Pair Problem:

**Input:** \( P \subseteq \mathbb{R}^d \)

**Output:** \( a, b \in P \) minimizing \( \| a - b \|_p \)

Colored Closest Pair Problem:

**Input:** \( A, B \subseteq \mathbb{R}^d \)

**Output:** \( (a, b) \in A \times B \) minimizing \( \| a - b \|_p \)
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Is **Colored** Closest Pair computationally **harder** than **Uncolored** Closest Pair?
Set Intersection

Uncolored \( k \)-Set Intersection Problem:

**Input:** \( S_1, \ldots, S_n \subseteq [n] \)

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**Input:** $S_1^1, \ldots, S_n^1, S_1^2, \ldots, S_n^2, \ldots, S_1^k, \ldots, S_n^k \subseteq [n]

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Is Colored $k$-Set Intersection problem computationally harder than Uncolored $k$-Set Intersection problem?
Using Color Coding we can reduce Uncolored version to Colored version
Big Question

Using *Color* Coding we can reduce *Uncolored* version to *Colored* version

Can we reduce *Colored* version to *Uncolored* version?
Outline of Talk

- Colored vs. Uncolored Problems ✓
- Closest Pair Problem
- Parameterized Set Intersection Problem
Closest Pair
Closest Pair problem (CP) in $\ell_p$-metric
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**Input:** $A \subset \mathbb{R}^d$, $|A| = n$
Closest Pair problem (CP) in $\ell_p$-metric

**Input:** $A \subset \mathbb{R}^d$, $|A| = n$

**Output:** $a^*, b^* \in A$, $\min_{a,b \in A, a \neq b} \|a - b\|_p$
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**Output:** $a^*, b^* \in A$, $\min_{a, b \in A, a \neq b} \|a - b\|_p$

**Trivial algorithm:** $O(n^2d)$
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- **Trivial algorithm:** $O(n^2d)$
- **Bently-Shamos’76:** $2^{O(d)}n \log n$ (for $\ell_2$-metric)
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  - Subcubic algorithms when $d = O(n)$ [ILLP04, MKZ09, GS17]
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- What happens when $d \approx \text{polylog } n$?
Bichromatic Closest Pair problem (BCP) in $\ell_p$ metric
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- **Computationally equivalent** to determining Minimum Spanning Tree in $\ell_p$-metric [AESW91, KLN99]
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- What happens when $d \approx \text{polylog } n$?
- What happens when $d = \omega(1)$?
Strong Exponential Time Hypothesis (SETH)

For every $\varepsilon > 0$, no algorithm running in $2^{m(1-\varepsilon)}$ time can solve $k$-SAT on $m$ variables.
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Let $p \geq 1$. Assuming SETH, for every $\varepsilon > 0$, no $n^{2-\varepsilon}$ time algorithm can solve:

- $\bigcirc$ BCP in $\ell_p$-metric when $d = \omega(\log n)$ [AW15]
Bichromatic Closest Pair under Fine-Grained Lens

Strong Exponential Time Hypothesis (SETH)

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- $\odot$ BCP in $\ell_p$-metric when $d = \omega(\log n)$ [AW15]

- $\odot$ $(1 + \delta)$-approximate BCP in $\ell_p$-metric when $d = \omega(\log n)$ [R18]
Let $p \geq 1$. Assuming SETH, for every $\varepsilon > 0$, no $n^{2-\varepsilon}$ time algorithm can solve: 

- BCP in $\ell_p$-metric when $d = \omega(\log n)$ [AW15]
- $(1 + \delta)$-approximate BCP in $\ell_p$-metric when $d = \omega(\log n)$ [R18]
- BCP in $\ell_p$-metric when $d = 2^{O(\log^* n)}$ [W18, C18]
BCP is at least as hard as CP in every $\ell_p$-metric for all $d$. 

Theorem (K-Manurangsi'18) ⊚ BCP and CP in $\ell_p$-metric are computationally equivalent when $3 = (\log d)^\Omega(1)$.

⊚ $(1 + \epsilon)$-approximate BCP can be solved by $\tilde{O}(\epsilon)$ calls to $(1 + \epsilon)$-approximate CP in $\ell_p$-metric when $3 = \epsilon$. 


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\[ \text{CP} \]
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Equivalence of Bichromatic Closest Pair and Closest Pair

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BCP and CP in $\ell_p$-metric are computationally equivalent when $3 = (\log =)$ $\Omega(1)$.

$\blacklozenge (1 + \varepsilon)$-approximate BCP can be solved by $\tilde{\Theta}(\sqrt{\varepsilon})$ calls to $(1 + \varepsilon)$-approximate CP in $\ell_p$-metric when $3 = (\log =)$. 

CP

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\text{CP}
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BCP is at least as hard as CP in every $\ell_p$-metric for all $d$.

Theorem (K-Manurangsi’18)

- BCP and CP in $\ell_p$-metric are computationally equivalent when $d = (\log n)^{\Omega(1)}$.
- $(1 + \delta)$-approximate BCP can be solved by $\tilde{O}(\sqrt{n})$ calls to $(1 + \delta)$-approximate CP in $\ell_p$-metric when $d = \omega(\log n)$. 
Every $(D_1, \ldots, D_i) \in \mathcal{P} \times \cdots \times \mathcal{P}$ has $C$ common neighbors in $\mathcal{P}$.

Every $\mathcal{F} \subset \mathcal{P} \cap \mathcal{P}$ has at most $C - 1$ common neighbors in $\mathcal{P}$, if $\mathcal{F} \cap \mathcal{P} = \emptyset$ for some $\mathcal{P}$ in $\mathcal{P}$.
Every \((u_1, \ldots, u_k)\) in \(U_1 \times \cdots U_k\) has \(t\) common neighbors in \(W\).
Every $(u_1, \ldots, u_k)$ in $U_1 \times \cdots \times U_k$ has $t$ common neighbors in $W$.

Every $X \subset U$ ($|X| = k$) has at most $t - 1$ common neighbors in $W$ if $X \cap U_i = \emptyset$ for some $i \in [k]$. 

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Panchromatic Graphs
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Every \(X \subset U\) \((|X| = k)\) has at most \(t - 1\) common neighbors in \(W\) if \(X \cap U_i = \emptyset\) for some \(i \in [k]\).

Do they exist?
Panchromatic Graphs when $k = 2$

Every $(D_1, D_2)$ in $\mathbb{D} \times \mathbb{D}$ has $C$ common neighbors in $\mathbb{H}$,

Every $\{D, D'\} \subset \mathbb{D}$ has at most $C - 1$ common neighbors in $\mathbb{H}$,
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Do they exist?
Bichromatic Closest Pair in \( \{0, 1\}^d \)
Bichromatic Closest Pair in $\{0, 1\}^d$

Points

Coordinates

Edge: $x \in A \cup B$ and $i \in [d]$ if $x_i = 1$
Bichromatic Closest Pair in $\{0,1\}^d$

Edge: $x \in A \cup B$ and $i \in [d]$ if $x_i = 1$

Minimizing Distance $\iff$
Maximizing Inner Product $\iff$
Maximizing Common Neighbors
Panchromatic Graph Composition

Points

Coordinates

\[ 3 \text{ copies} / \text{one.taboldstyle/eight.taboldstyle} \]
Panchromatic Graph Composition

Points

Coordinates

A

B

[d]

W

d copies
Panchromatic Graphs when $k = 2$ [K-Manurangsi’18]

Many $(u_1, u_2)$ in $U_1 \times U_2$ has $t$ common neighbors in $W$

Every $\{u, u'\} \subset U_i$ has at most $t - 1$ common neighbors in $W$
Construction of Panchromatic graphs when $k = 2$

Polynomials are our friends.

– TCS Folklore
Construction of Panchromatic graphs when $k = 2$

- $U_1 :=$ set of degree $d$ univariate polynomials over $\mathbb{F}_q$
Construction of Panchromatic graphs when $k = 2$

- $U_1 := \text{set of degree } d \text{ univariate polynomials over } \mathbb{F}_q$
- $U_2 := \{x^{d+1} + p(x) \mid p(x) \in U_1\}$
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- $U_1 :=$ set of degree $d$ univariate polynomials over $\mathbb{F}_q$
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- $U_1 := \text{set of degree } d \text{ univariate polynomials over } \mathbb{F}_q$
- $U_2 := \{x^{d+1} + p(x) \mid p(x) \in U_1\}$
- $W = \mathbb{F}_q \times \mathbb{F}_q$
- $(p, (\alpha, \beta)) \in U \times W$ is an edge $\iff p(\alpha) = \beta$

Polynomials are our friends. – TCS Folklore
Panchromatic Graphs when $k = 2$

They exist! $(?, ?') \in \mathbb{Z}_2^8$ have $(?, ?')$ as common neighbor $\Rightarrow$ is root of $? - ?'$ $\Rightarrow (?, ?') \in \mathbb{Z}_2^8$ have at most 3 common neighbors $(?, G_{3+1} + ?') \in \mathbb{Z}_1^2 \times \mathbb{Z}_2^2$ have $3+1$ common neighbors $\Leftrightarrow G_{3+1} + ?' - ?'$ has $3+1$ distinct roots

Number of such polynomials: $(\mathbb{F}_q \times \mathbb{F}_q)$
Panchromatic Graphs when $k = 2$

$\exists F \times F$ as common neighbor

$(p, p') \in U_i$ have $(\alpha, \beta)$

$\Rightarrow \alpha$ is root of $p - p'$

$\Rightarrow (p, p') \in U_i$ have at most $d$ common neighbors

Polynomials

$F_q \times F_q$
Panchromatic Graphs when \( k = 2 \)

\[(p, p') \in U_i \text{ have } (\alpha, \beta) \text{ as common neighbor} \]

\[\Rightarrow \alpha \text{ is root of } p - p' \]

\[\Rightarrow (p, p') \in U_i \text{ have at most } d \text{ common neighbors} \]

\[(p, x^{d+1} + p') \in U_1 \times U_2 \]

have \( d + 1 \) common neighbors

\[\Leftrightarrow x^{d+1} + p' - p' \text{ has } d + 1 \text{ distinct roots} \]

Number of such polynomials: \( \binom{q}{d+1} \)
Panchromatic Graphs when \( k = 2 \)

Let \( \mathbb{F}_q \times \mathbb{F}_q \) be a set of polynomials. If \( (p, p') \in U_i \) have \((\alpha, \beta)\) as common neighbor, then \( \alpha \) is root of \( p - p' \). This implies \( (p, p') \in U_i \) have at most \( d \) common neighbors.

Let \( (p, x^{d+1} + p') \in U_1 \times U_2 \) have \( d + 1 \) common neighbors, which means \( x^{d+1} + p' - p' \) has \( d + 1 \) distinct roots.

Number of such polynomials: \( \binom{q}{d+1} \)

They exist!
Theorem (K-Manurangsi’18)

- BCP and CP in $\ell_p$-metric are computationally equivalent when $d = (\log n)^{\Omega(1)}$.

- $(1 + \delta)$-approximate BCP can be solved by $\tilde{O}(\sqrt{n})$ calls to $(1 + \delta)$-approximate CP in $\ell_p$-metric when $d = \omega(\log n)$. 

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- **Colored** vs. **Uncolored Problems** ✔
- Closest Pair Problem ✔
- Parameterized Set Intersection Problem
Set Intersection
**k-Set Intersection**

**Input:** \( S_1, \ldots, S_n \subseteq [n] \)

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- NP World: Ruling out PTAS (assuming NP $\neq$ P) is open!
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- NP World: Ruling out PTAS (assuming \( \text{NP} \neq \text{P} \)) is open!
- No poly factor approximation poly time algorithm assuming "weak-ETH" [Xavier'12]
  - Relies on Quasi-random PCP of [Khot’06]
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- $W[1]$$\neq$FPT: No $F(k)$ factor approximation $T(k)\cdot\text{poly}(n)$ time algorithm [Lin’15]
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- **W[1]$\neq$FPT:** No $F(k)$ factor approximation $T(k) \cdot \text{poly}(n)$ time algorithm [Lin’15]
- **ETH:** No $F(k)$ factor approximation $n^{\Omega(\sqrt{k})}$ time algorithm [Lin’15]
Colored $k$-Set Intersection

**Input:** $S_1^1, \ldots, S_n^1, S_1^2, \ldots, S_n^2, \ldots, S_1^k, \ldots, S_n^k \subseteq [n]$

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◎ NP World: Essentially same as Extended Label Cover
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- NP World: Essentially same as Extended Label Cover
- $W[1] \neq \text{FPT}$: No $F(k)$ factor approximation $T(k) \cdot \text{poly}(n)$ time algorithm [K-Laekhanukit-Manurangsi’18]
Colored $k$-Set Intersection

**Input:** $S^1_1, \ldots, S^1_n, S^2_1, \ldots, S^2_n, \ldots, S^k_1, \ldots, S^k_n \subseteq [n]$

**Output:** $S^1_{i_1}, \ldots, S^k_{i_k}$ whose intersection is maximized

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- Tight running time lower bounds under $W[1] \neq \text{FPT}$, ETH, and SETH for exact version are straightforward!
Our Result: Equivalence

Theorem (Bukh-K-Narayanan’21)

- $k$-Set Intersection and Colored $k$-Set Intersection are computationally equivalent up to $O_k(1)$ factors in run time.
Our Result: Equivalence

Theorem (Bukh-K-Narayanan’21)

- $k$-Set Intersection and Colored $k$-Set Intersection are computationally equivalent up to $O_k(1)$ factors in run time.

- $c$-approximation of $k$-Set Intersection is harder than $c/h(k)$-approximation of Colored $k$-Set Intersection.
They exist! Many \((D_1, \ldots, D_m)\) in \(*_1 \times \cdots \times *_{m'}\) has \(C\) common neighbors in \(*\), every \(- \subseteq *_{m'}\) has at most \(C/2\) common neighbors in \(*\), if \(- \cap *_{m'} = \emptyset\) for some \(8\in\[\]\).
Many \((u_1, \ldots, u_k)\) in \(U_1 \times \cdots U_k\) has \(t\) common neighbors in \(W\)
Many \((u_1, \ldots, u_k)\) in \(U_1 \times \cdots U_k\) has \(t\) common neighbors in \(W\).

Every \(X \subset U\) (\(|X| = k\)) has at most \(t/F(k)\) common neighbors in \(W\) if \(X \cap U_i = \emptyset\) for some \(i \in [k]\).
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Every \(X \subset U\) (\(|X| = k\)) has at most \(t/F(k)\) common neighbors in \(W\) if \(X \cap U_i = \emptyset\) for some \(i \in [k]\).

They exist!
Set Intersection Lower Bounds

- \( W[1] \neq \text{FPT} \): No \( F(k) \) factor approximation \( T(k) \cdot \text{poly}(n) \) time algorithm [Lin’15]

- \( \text{ETH} \): No \( F(k) \) factor approximation \( n^{\Omega(k)} \) time algorithm [Bukh-K-Narayanan’21]

- \( \text{SETH} \): No \( F(k) \) factor approximation \( n^{k-\epsilon} \) time algorithm [Bukh-K-Narayanan’21]
Colored $k$-Set Intersection Problem

\[ C_i = \{S_1^i, \ldots, S_n^i\} \]
Panchromatic Graph Composition
Panchromatic Graph Composition
Edge between $S^j_i$ and $(a, w) \iff a \in S^j_i$ and edge between $S^j_i$ and $w$ in Panchromatic Graph
Polynomials are still our friends.

– TCS Folklore
Construction of Panchromatic graphs

Pick \( w_1, \ldots, w_k \) random \( k \)-variate polynomials over \( \mathbb{F}_q \) of degree at most \( D \).
Pick $w_1, \ldots, w_k$ random $k$-variate polynomials over $\mathbb{F}_q$ of degree at most $D$

$U_i^0$ is a set of $n$ random $k$-variate polynomials over $\mathbb{F}_q$ of degree at most $d$

$U_i := w_i + U_i^0$
Construction of Panchromatic graphs

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$(p + w_i, \alpha) \in U \times W$ is an edge $\iff$ $\alpha$ is a root of $p + w_i$
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- $U_i := w_i + U^0_i$

- $W = \mathbb{F}_q^k$

- $(p + w_i, \alpha) \in U \times W$ is an edge $\Leftrightarrow \alpha$ is a root of $p + w_i$

- $w_i + p$ is uniform on $\mathbb{F}_q[X_1, \ldots, X_k]_{\leq D}$
Theorem (Bukh-K-Narayanan’21)

For $k, d \in \mathbb{N}$ and a prime power $q \in \mathbb{N}$, let $Z$ be the (random) number of common roots over $\mathbb{F}_q^k$ of $k$ independently chosen $k$-variate random $\mathbb{F}_q$-polynomials of degree $d$. Then, as $q \to \infty$, we have
Theorem (Bukh-K-Narayanan’21)

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$$\Pr[Z = d^k] \geq \frac{1 - o(1)}{(d^k)!},$$
Theorem (Bukh-K-Narayanan’21)

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\[
\Pr[Z = d^k] \geq \frac{1 - o(1)}{(d^k)!},
\]
as well as

\[
\Pr[Z > d^k] = O(q^{-d}).
\]
Fix $S = \{w_i + p_i \in U_i | i \in [k]\}$
Fix $S = \{ w_i + p_i \in U_i | i \in [k] \}$

- $|N(S)|$ is distributed as the number of $\mathbb{F}_q$-solutions of $k$ random polynomials from $\mathbb{F}_q[X_1, \ldots, X_k]_{\leq D}$
Analysis of Construction

Fix $S = \{w_i + p_i \in U_i | i \in [k]\}$

- $|N(S)|$ is distributed as the number of $\mathbb{F}_q$-solutions of $k$ random polynomials from $\mathbb{F}_q[X_1, \ldots, X_k]_{\leq D}$

- $\Pr[|N(S)| > D^k] = O(q^{-D})$
  - By parameter choice, number of such sets $< 1/q$
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Fix $S \subseteq U, |S| = k$ and $S \cap U_1 = \emptyset$. 
Analysis of Construction

Fix $S = \{w_i + p_i \in U_i | i \in [k]\}$

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- $|N(S)|$ is distributed as the number of $\mathbb{F}_q$-solutions of $k$ random polynomials from $\mathbb{F}_q[X_1, \ldots, X_k]_{\leq D}$ or $\mathbb{F}_q[X_1, \ldots, X_k]_{\leq d}$
Fix $S = \{w_i + p_i \in U_i | i \in [k]\}$

- $|N(S)|$ is distributed as the number of $\mathbb{F}_q$-solutions of $k$ random polynomials from $\mathbb{F}_q[X_1, \ldots, X_k]_{\leq D}$
- $\Pr[|N(S)| > D^k] = O(q^{-D})$
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- $\Pr[|N(S)| = D^k] = \frac{1}{(D^k)!}$

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- $\Pr[|N(S)| > dD^{k-1}] = O(q^{-d})$
Many \((u_1, \ldots, u_k)\) in \(U_1 \times \cdots \times U_k\) has \(t\) common neighbors in \(W\).

Every \(X \subset U\) (\(|X| = k\)) has at most \(t/F(k)\) common neighbors in \(W\) if \(X \cap U_i = \emptyset\) for some \(i \in [k]\).

They exist!


- $W[1] \neq \text{FPT}$: No $F(k)$ factor approximation $T(k) \cdot \text{poly}(n)$ time algorithm [Lin’15]

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Set Intersection Lower Bounds

- **W[1]≠FPT:** No $F(k)$ factor approximation $T(k)\cdot\text{poly}(n)$ time algorithm [Lin’15] **New Proof!**

- **ETH:** No $F(k)$ factor approximation $n^{\Omega(k)}$ time algorithm [Bukh-K-Narayanan’21]

- **SETH:** No $F(k)$ factor approximation $n^{k-\varepsilon}$ time algorithm [Bukh-K-Narayanan’21]
Starting from \( k \)-Clique

\[ V = [n] \]

\[ E \]

Input: \( G([n], E) \)
Starting from $k$-Clique

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If $G$ has a $k$-clique then there are $\binom{k}{2}$ vertices in $E$ which in total have $k$ neighbors.
Starting from $k$-Clique

Input: $G([n], E)$

If $G$ has a $k$-clique then there are $\binom{k}{2}$ vertices in $E$ which in total have $k$ neighbors.

If $G$ has no $k$-clique then any $\binom{k}{2}$ vertices in $E$ has totally at least $k + 1$ neighbors.
Threshold Graph

\[ A = [n] \]

\[ V = [n] \]

Every vertex in \( A \) has at least \( \Omega \left( \frac{1}{\ell} \right) \) common neighbors in \( V \).

Every \( v \in V \) has at most \( \ell \) common neighbors in \( A \).
Threshold Graph

Every $k$ vertices in $V$ has at least $n^{\Omega(1/k)}$ common neighbors in $A$.
Threshold Graph

$$A = [n]$$

$$V = [n]$$

Every $k$ vertices in $V$ has at least $n^{\Omega(1/k)}$ common neighbors in $A$

Every $k + 1$ vertices in $V$ has at most $k^{O(k)}$ common neighbors in $A$
Threshold Graph Composition

\[ A = [n] \]

\[ V = [n] \]

\[ E \]

\((4,0) \in \times \) is an edge \( \iff \exists E, E' \in + \) such that 0 and 4 are common neighbors of \( E \) and \( E' \)
Threshold Graph Composition

\[ A = [n] \]

\[ V = [n] \]

\[ E \]

\[ (e, a) \in E \times A \text{ is an edge } \iff \exists v, v' \in V \text{ such that } \]

\[ a \text{ and } e \text{ are common neighbors of } v \text{ and } v' \]
Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of $k$-clique in $G$

- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph
Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of $k$-clique in $G$

- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph

- Every $a \in A'$ is also a common neighbor of $e_{v_i,v_j} \in E$
Completeness of Reduction

- Let \( v_1, \ldots, v_k \in V \) be vertices of \( k\)-clique in \( G \).
- Let \( A' \subseteq A \) be common neighbors of \( v_1, \ldots, v_k \) in Threshold graph.
- Every \( a \in A' \) is also a common neighbor of \( e_{v_i,v_j} \in E \).

Completeness of Threshold Graph

Every \( k \) vertices in \( V \) has at least \( n^{\Omega(1/k)} \) common neighbors in \( A \).
Soundness of Reduction

Fix \((e_1, \ldots, e_{\binom{k}{2}}) \in E\) and let \(A' \subseteq A\) be its set of common neighbors.
Soundness of Reduction

- Fix \((e_1, \ldots, e_{\binom{k}{2}}) \in E\) and let \(A' \subseteq A\) be its set of common neighbors.
- Let \(V' \subseteq V\) be set of total neighbors of \((e_1, \ldots, e_{\binom{k}{2}})\) in \(V\).
- \(|V'| \geq k + 1\)
Soundness of Reduction

- Fix \((e_1, \ldots, e_{k(2)}) \in E\) and let \(A' \subseteq A\) be its set of common neighbors.
- Let \(V' \subseteq V\) be set of total neighbors of \((e_1, \ldots, e_{k(2)})\) in \(V\).
- \(|V'| \geq k + 1\)
- \(A'\) is a subset of the common neighbors of \(V'\) in Threshold graph.
Soundness of Reduction

- Fix \((e_1, \ldots, e_{\binom{k}{2}}) \in E\) and let \(A' \subseteq A\) be its set of common neighbors.
- Let \(V' \subseteq V\) be set of total neighbors of \((e_1, \ldots, e_{\binom{k}{2}})\) in \(V\).
- \(|V'| \geq k + 1\).
- \(A'\) is a subset of the common neighbors of \(V'\) in Threshold graph.

Soundness of Threshold Graph

Every \(k + 1\) vertices in \(V\) has at most \(k^{O(k)}\) common neighbors in \(A\).
Threshold Graph

\[ A = \{1, 2, \ldots, n\} \]
\[ V = \{1, 2, \ldots, n\} \]

Every \( k \) vertices in \( V \) has at least \( n^{\Omega(1/k)} \) common neighbors in \( A \)

Every \( k + 1 \) vertices in \( V \) has at most \( k^{O(k)} \) common neighbors in \( A \)
Threshold Graph

\[ A = [n] \]

\[ V = [n] \]

Every vertex in \( V \) has at least \( \Omega \left( \frac{1}{\epsilon} \right) \) common neighbors in \( A \).

Every \( \epsilon + 1 \) vertices in \( V \) has at most \( \epsilon \) common neighbors in \( A \).
Outline of Talk

- Colored vs. Uncolored Problems ✓
- Closest Pair Problem ✓
- Parameterized Set Intersection Problem ✓
Key Takeaways

- Panchromatic Graphs Exist!
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- **Panchromatic** Graphs Exist!
- **Tight** Running Time Lower Bounds for *Approximating* Parameterized Set Intersection
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- Can we find explicit Panchromatic Graphs?
Key Takeaways

- **Panchromatic** Graphs Exist!
- **Tight** Running Time Lower Bounds for *Approximating* Parameterized Set Intersection
- Can we find **explicit** Panchromatic Graphs?
- Are there more **applications** for these graphs?
THANK YOU!