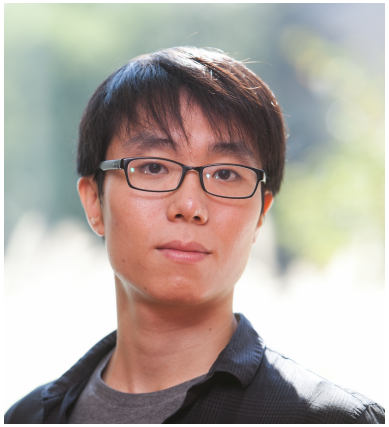
Clustering

How hard is it to Classify Data?

Karthik C. S.
(Tel Aviv University)

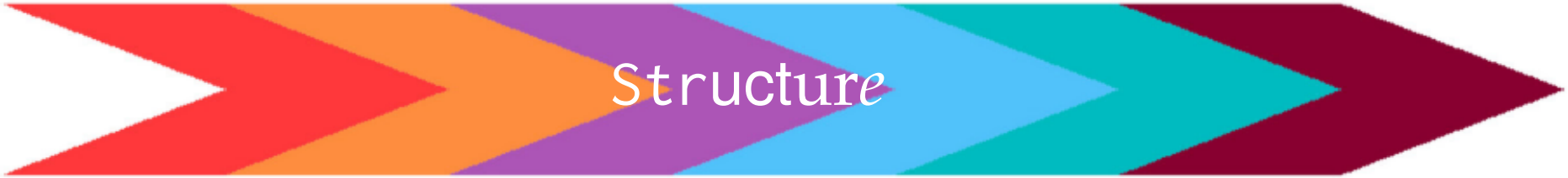
Joint work(s) with

Vincent Cohen-Addad
(Sorbonne University)



Euiwoong Lee
(New York University)

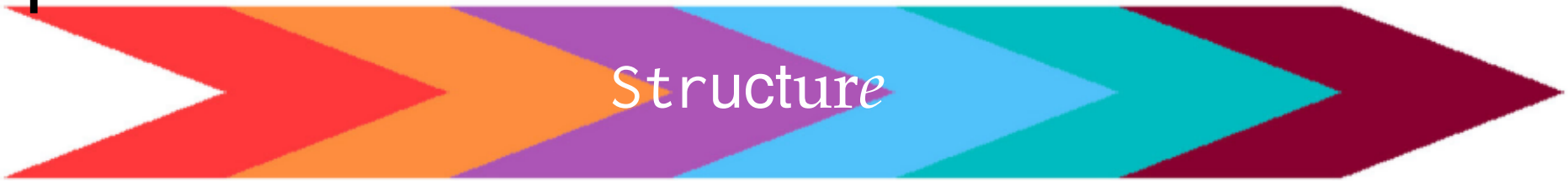
Spectrum of Computational Problems



Spectrum of Computational Problems

Circuit-SAT

CSP



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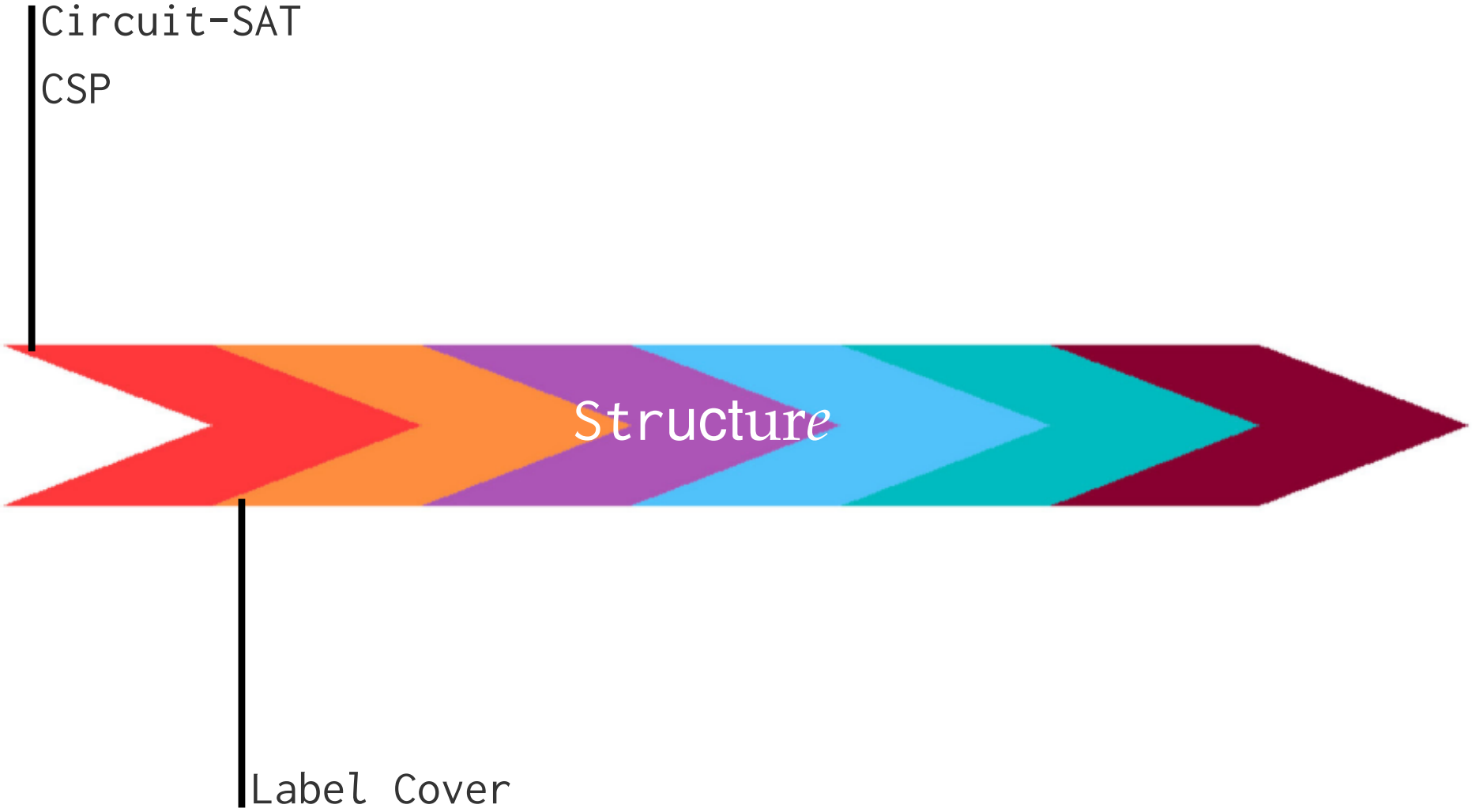

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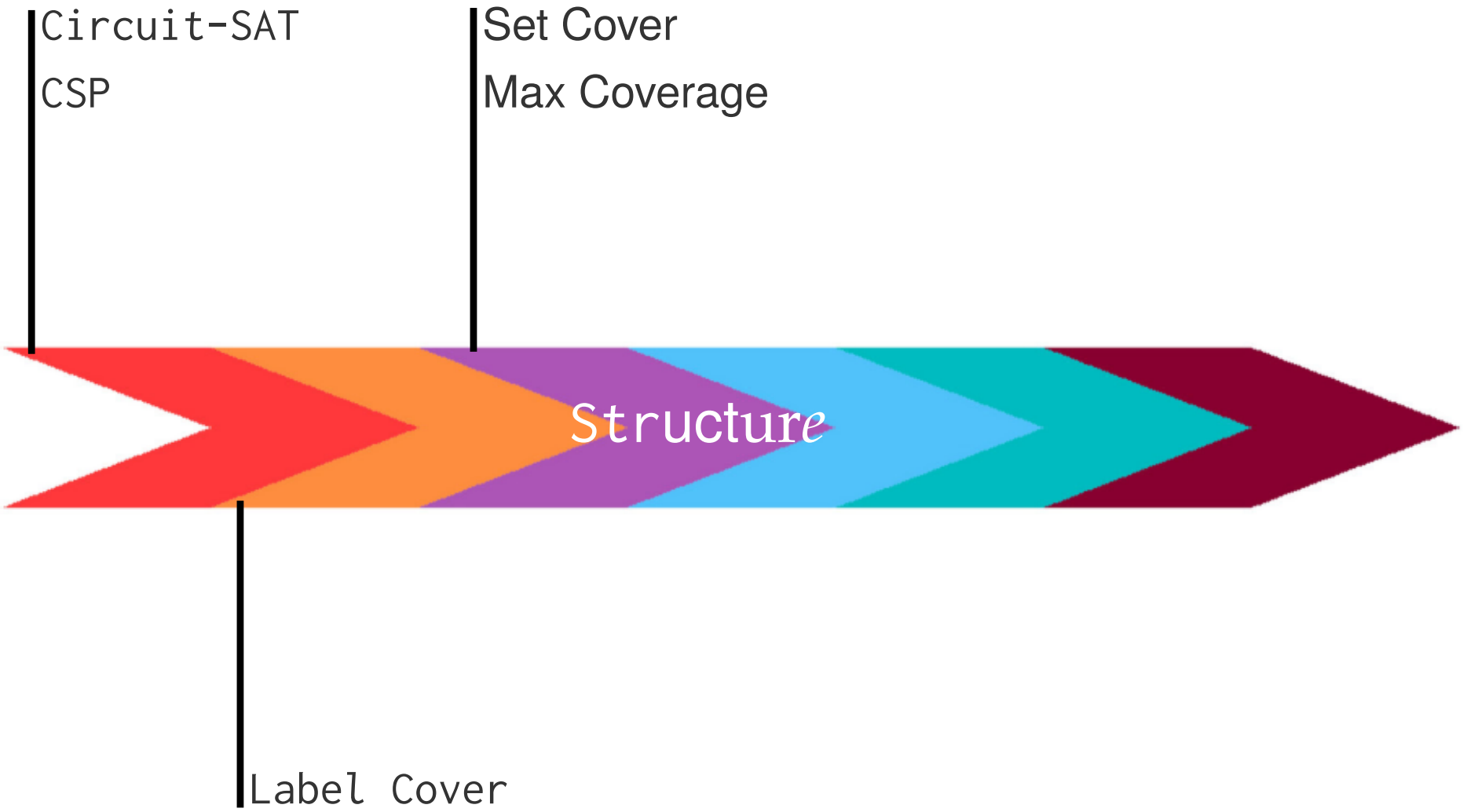

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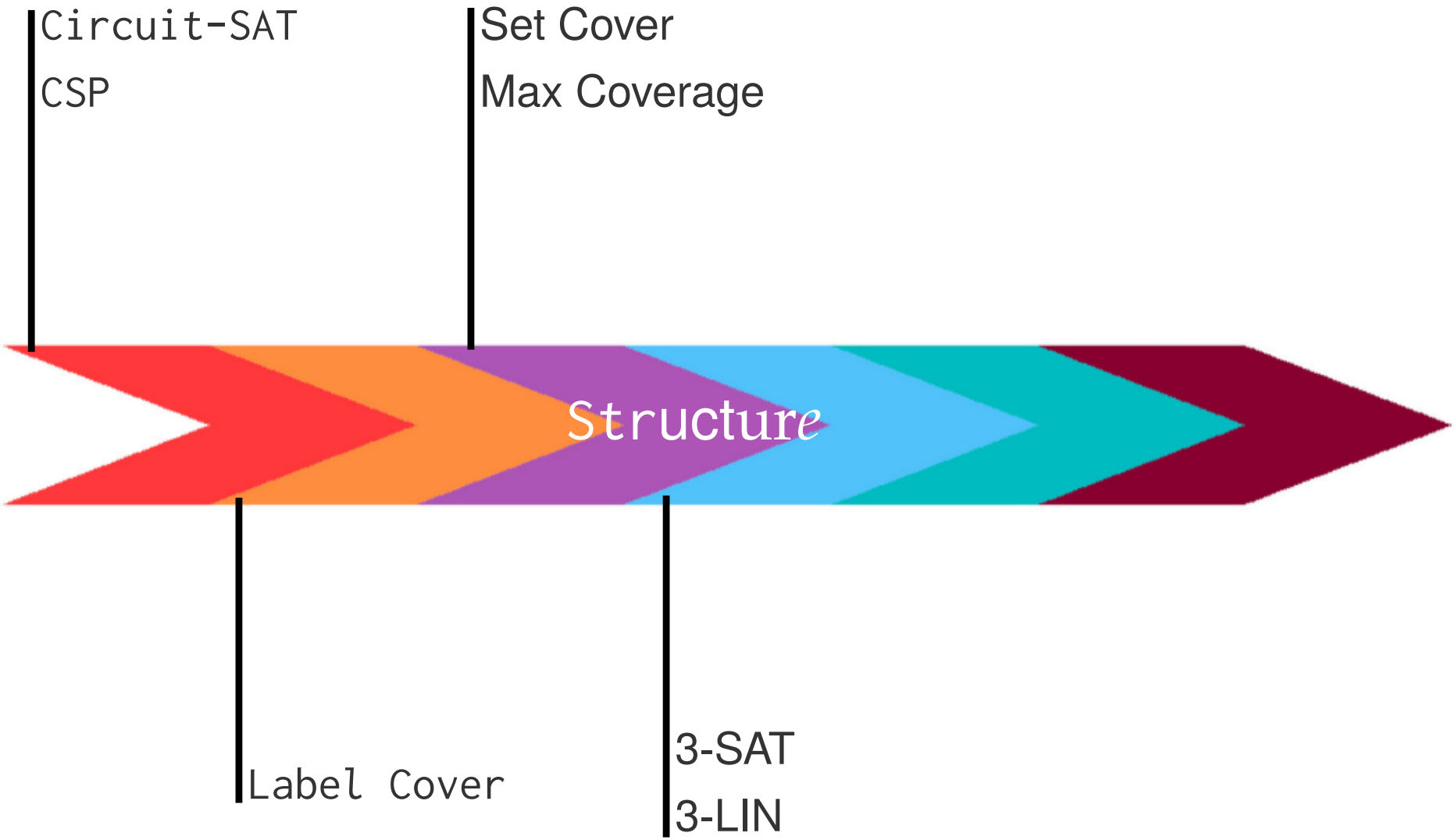

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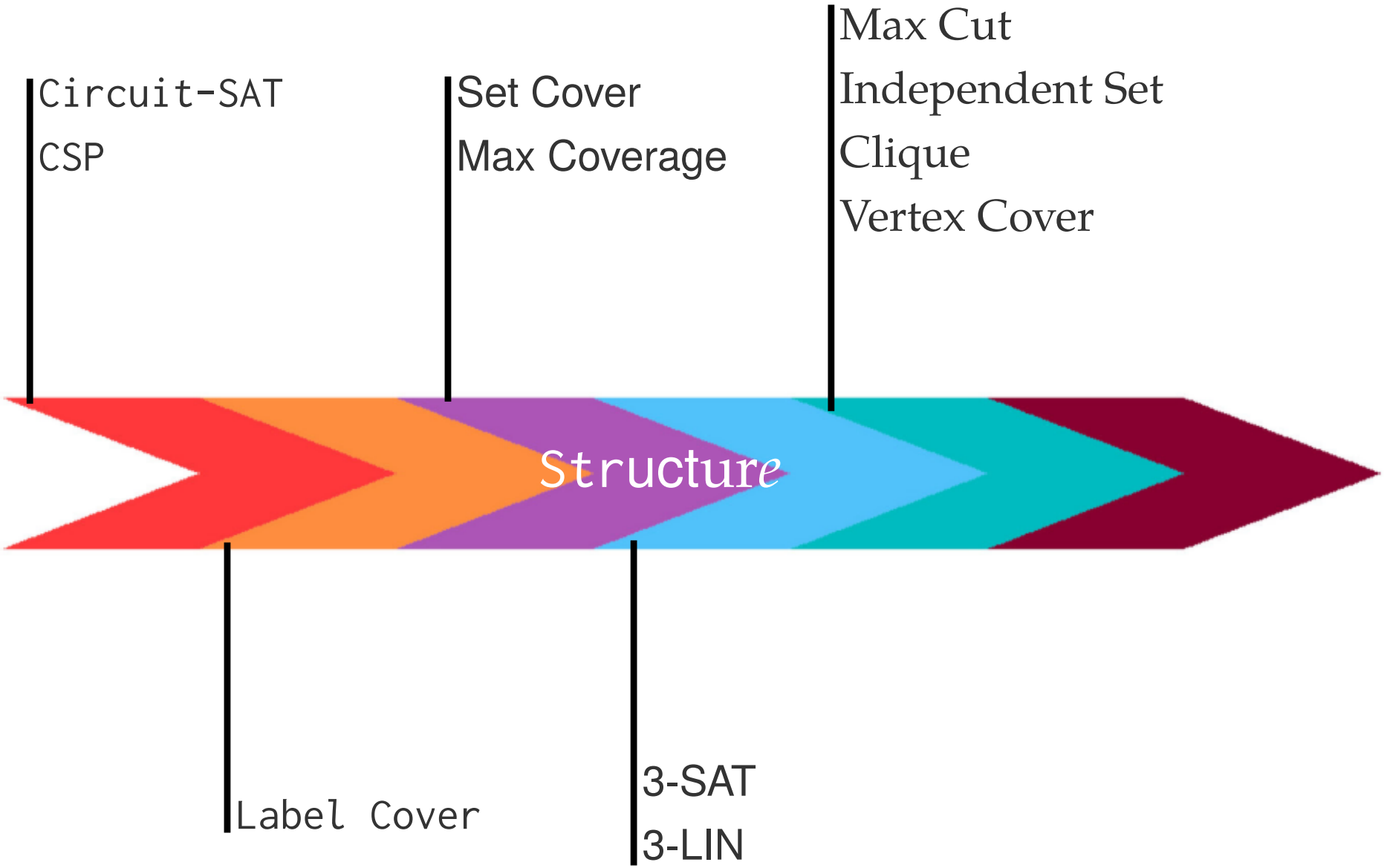

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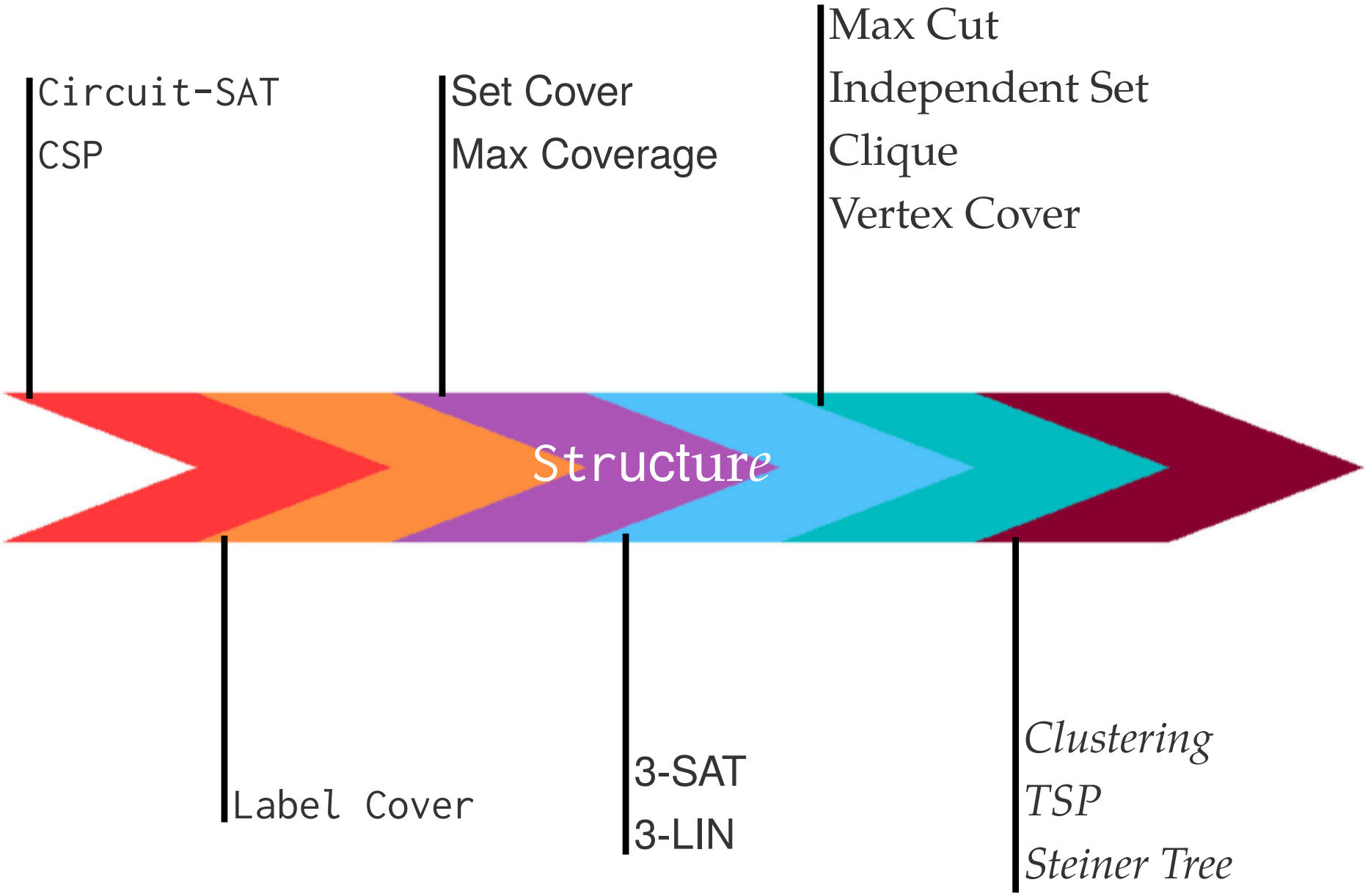

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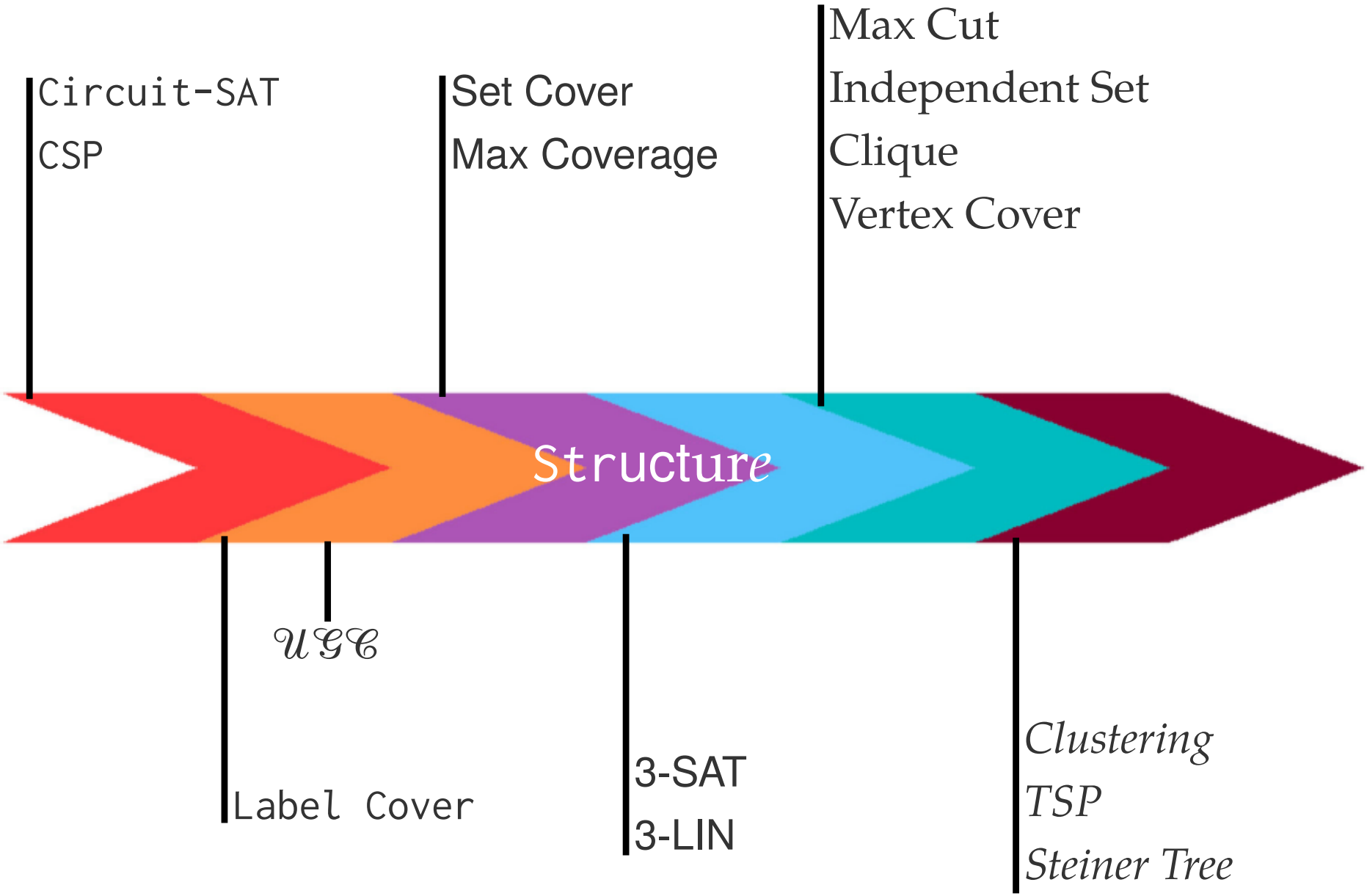

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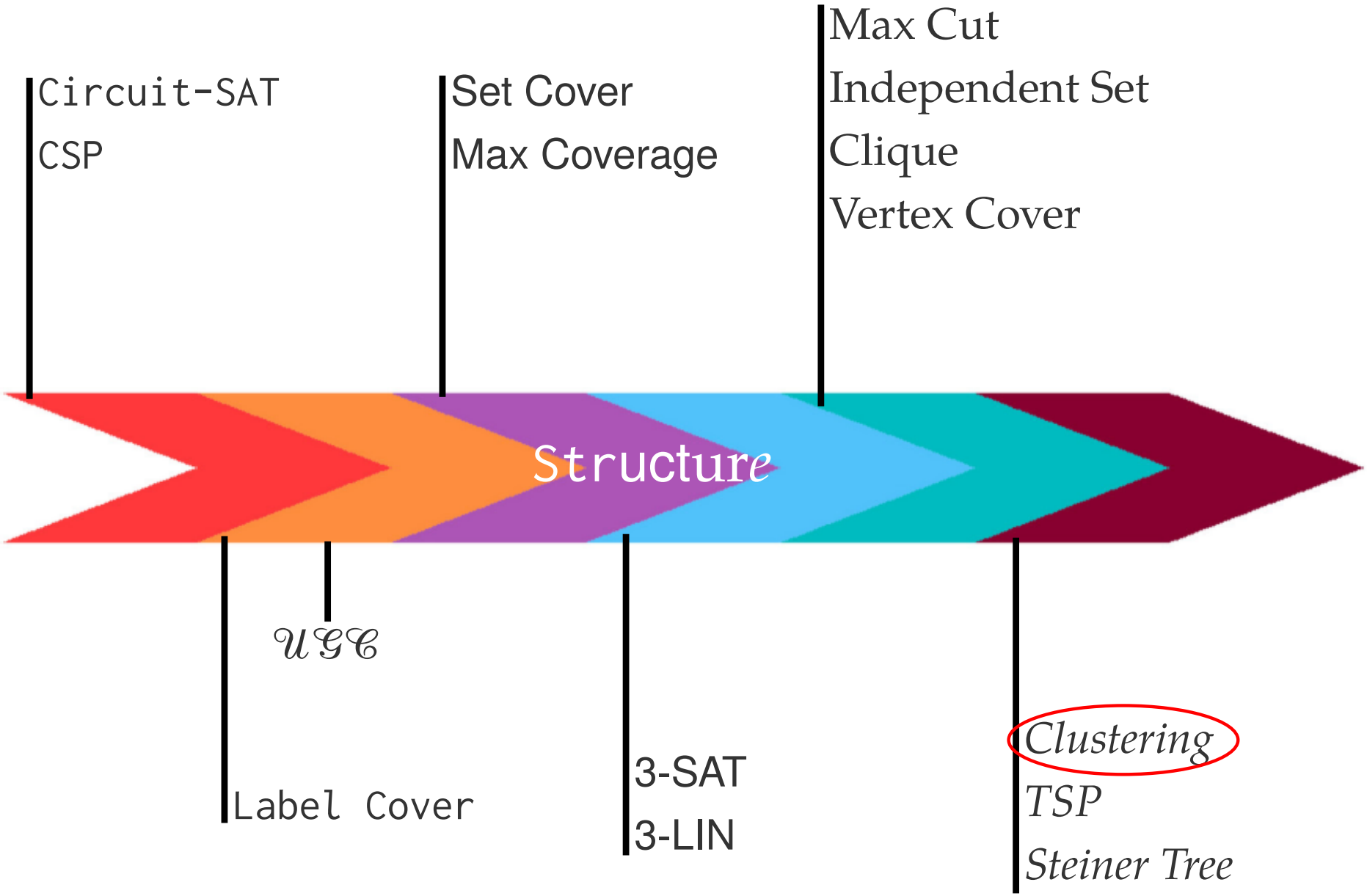

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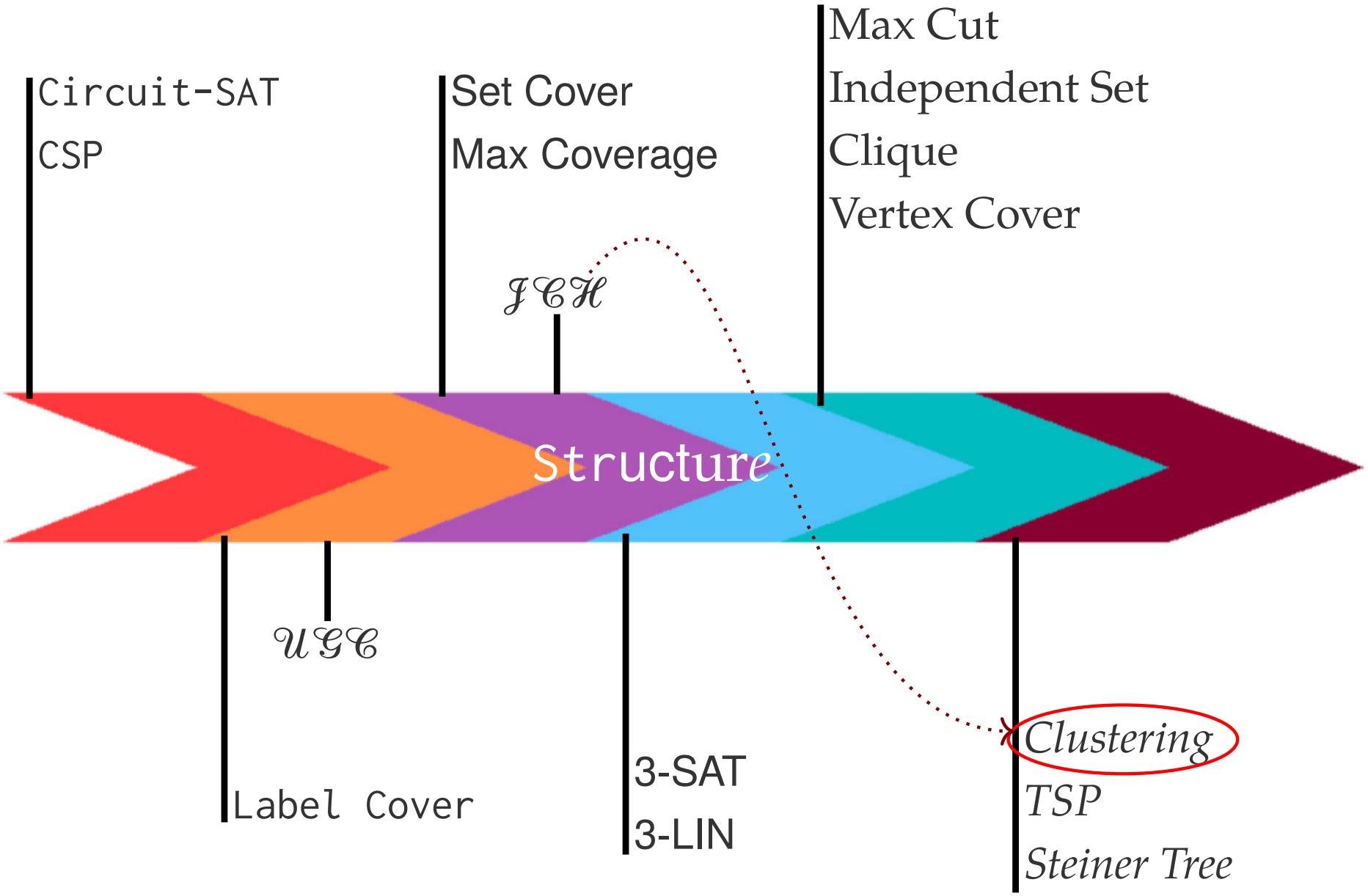

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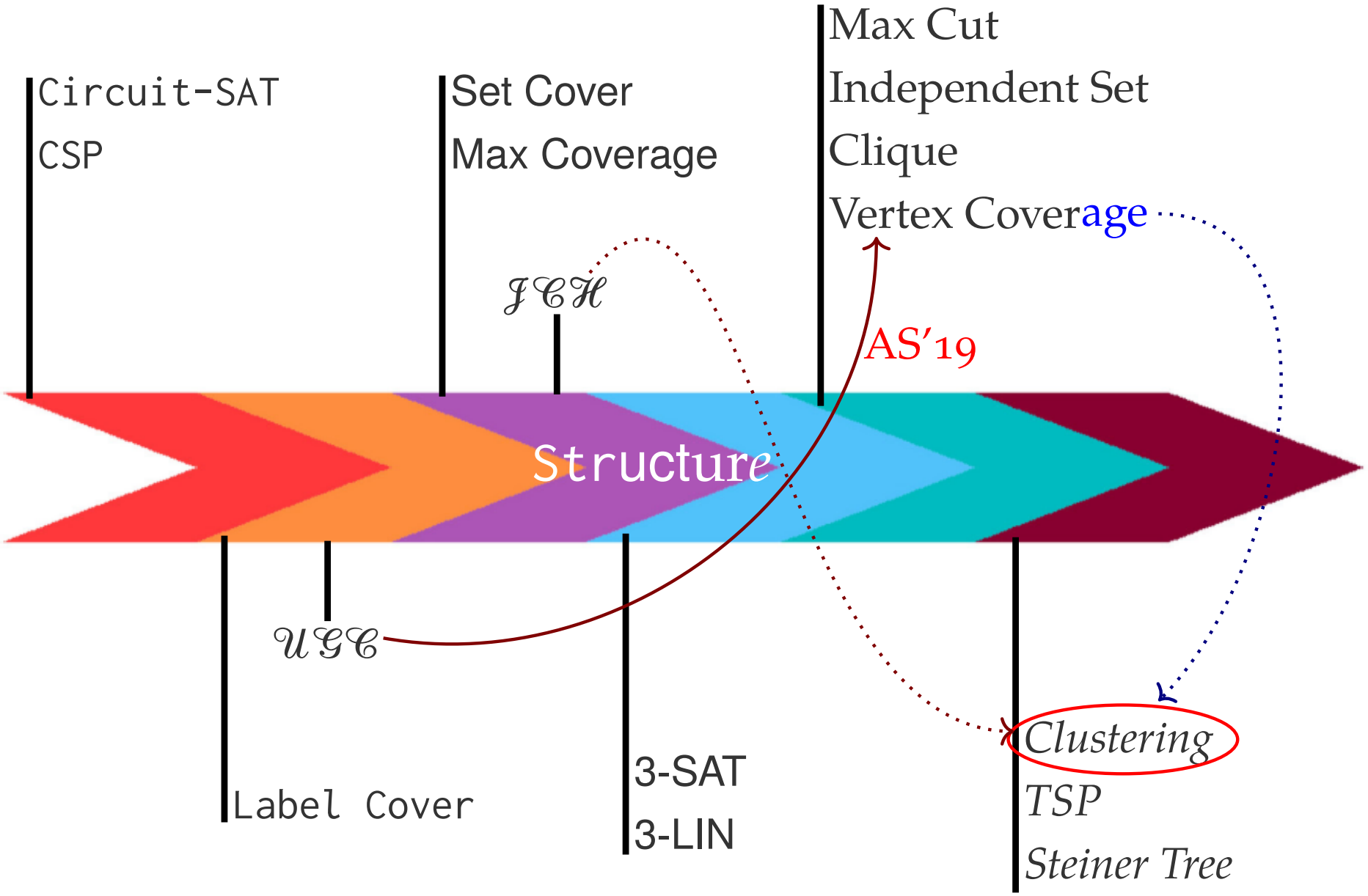

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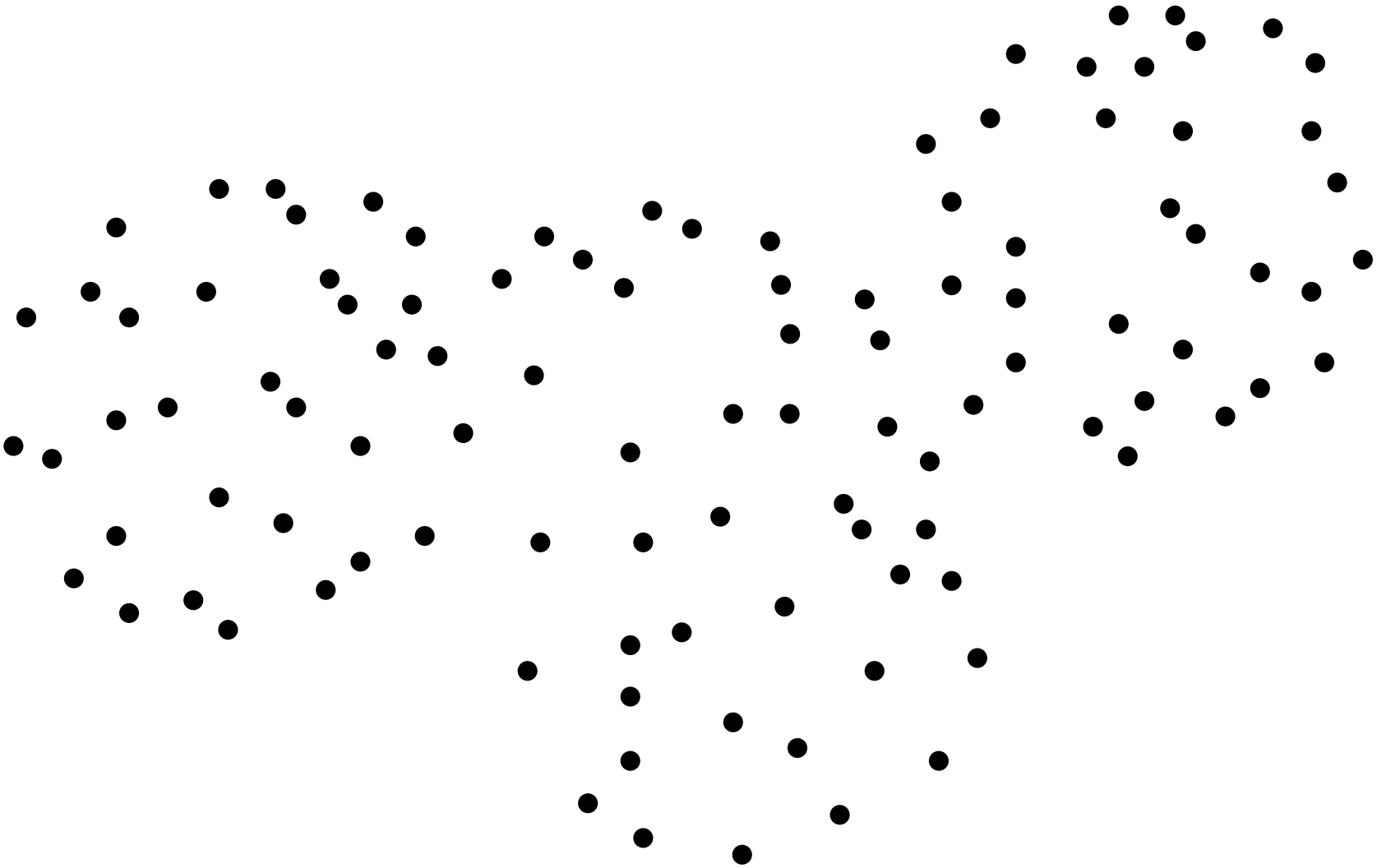

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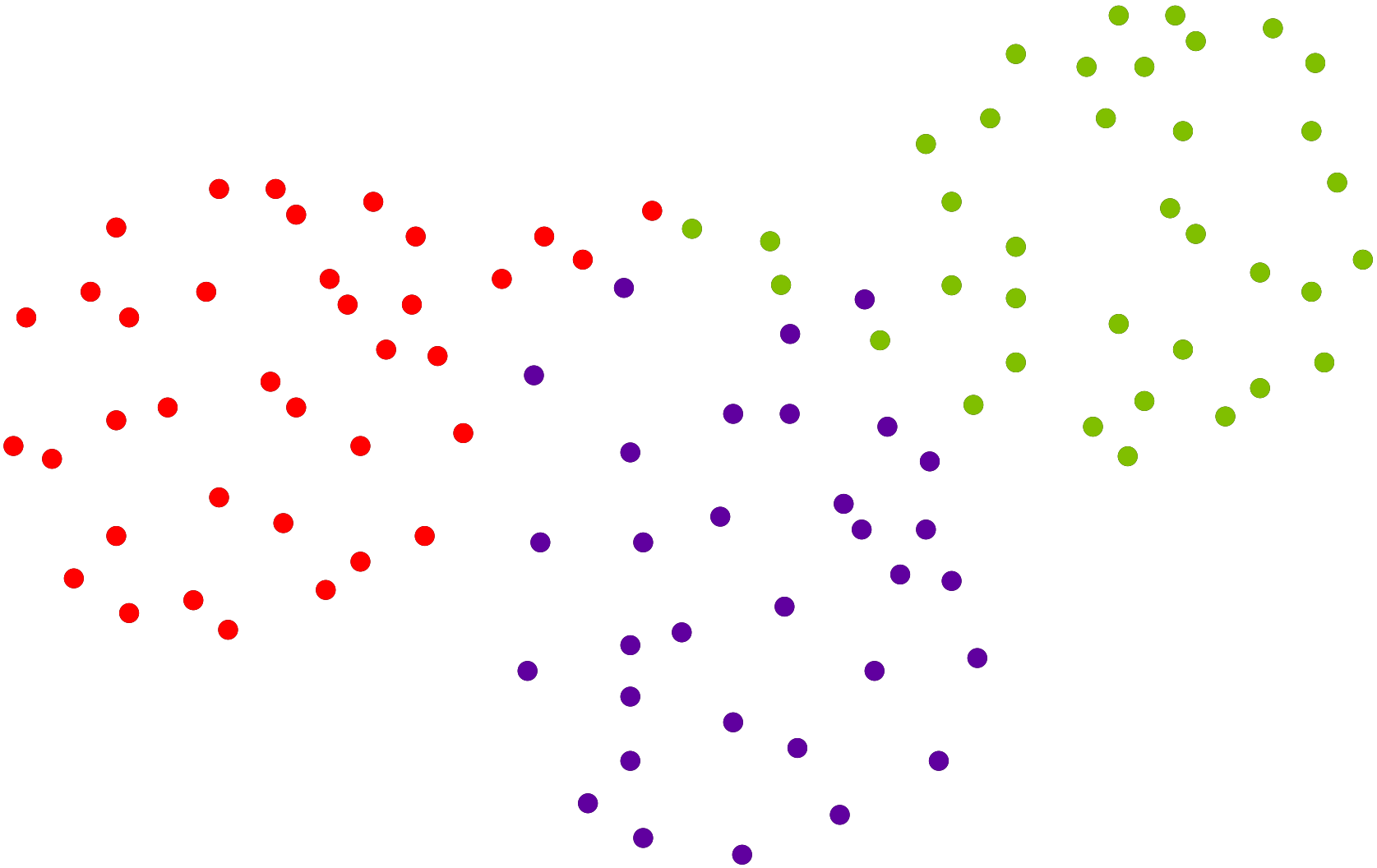

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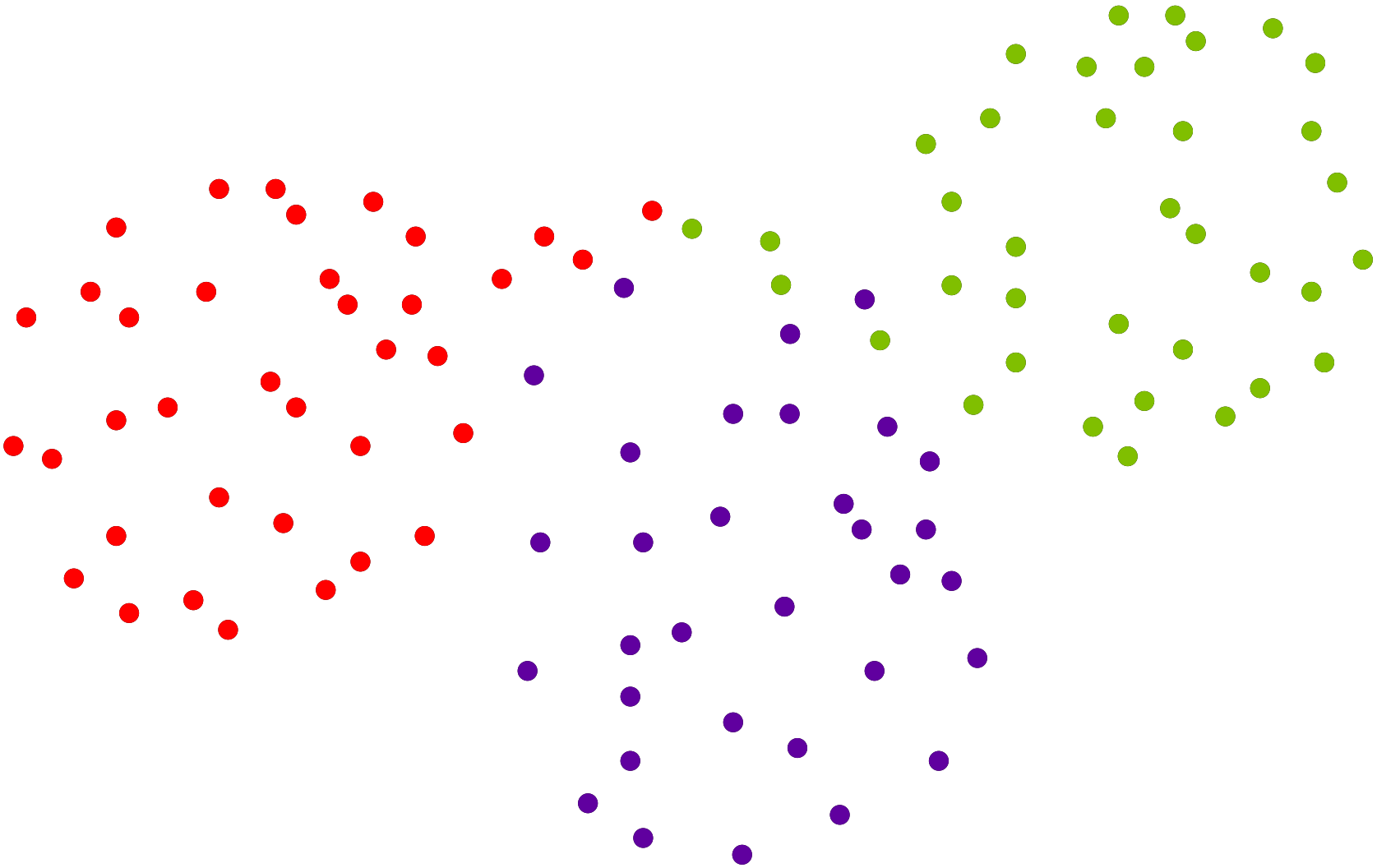

Task of Classifying Input Data

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A New Embedding Framework to potentially get Strong (tight?) Inapproximability results!

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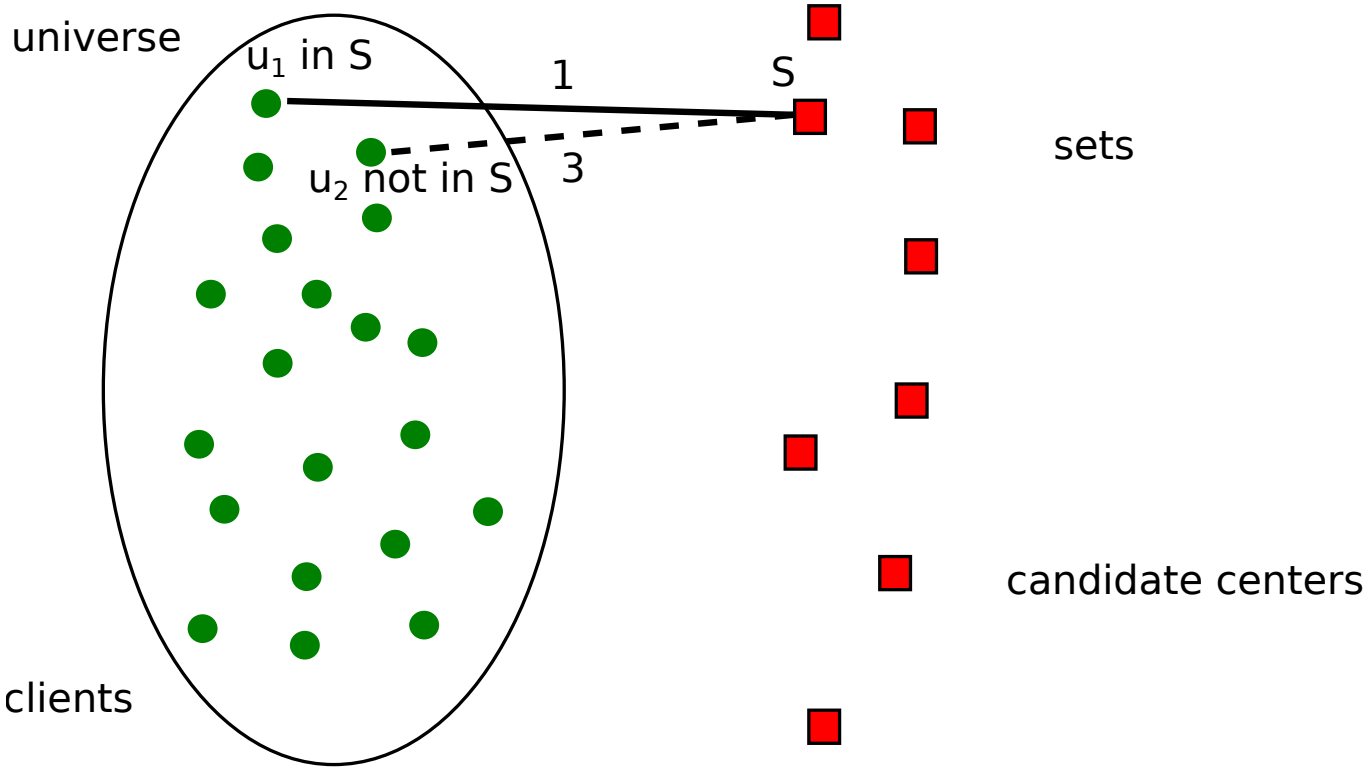

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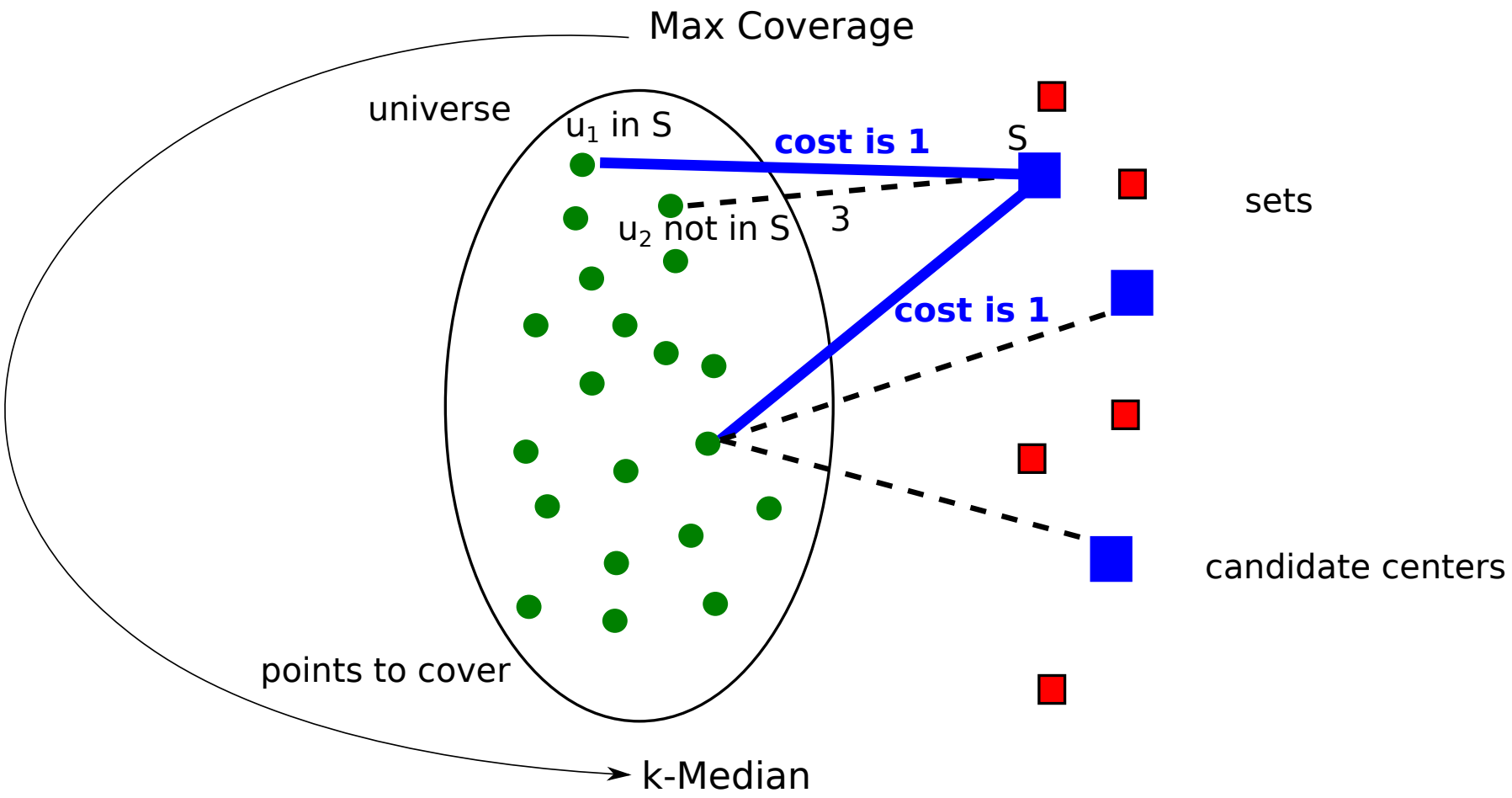

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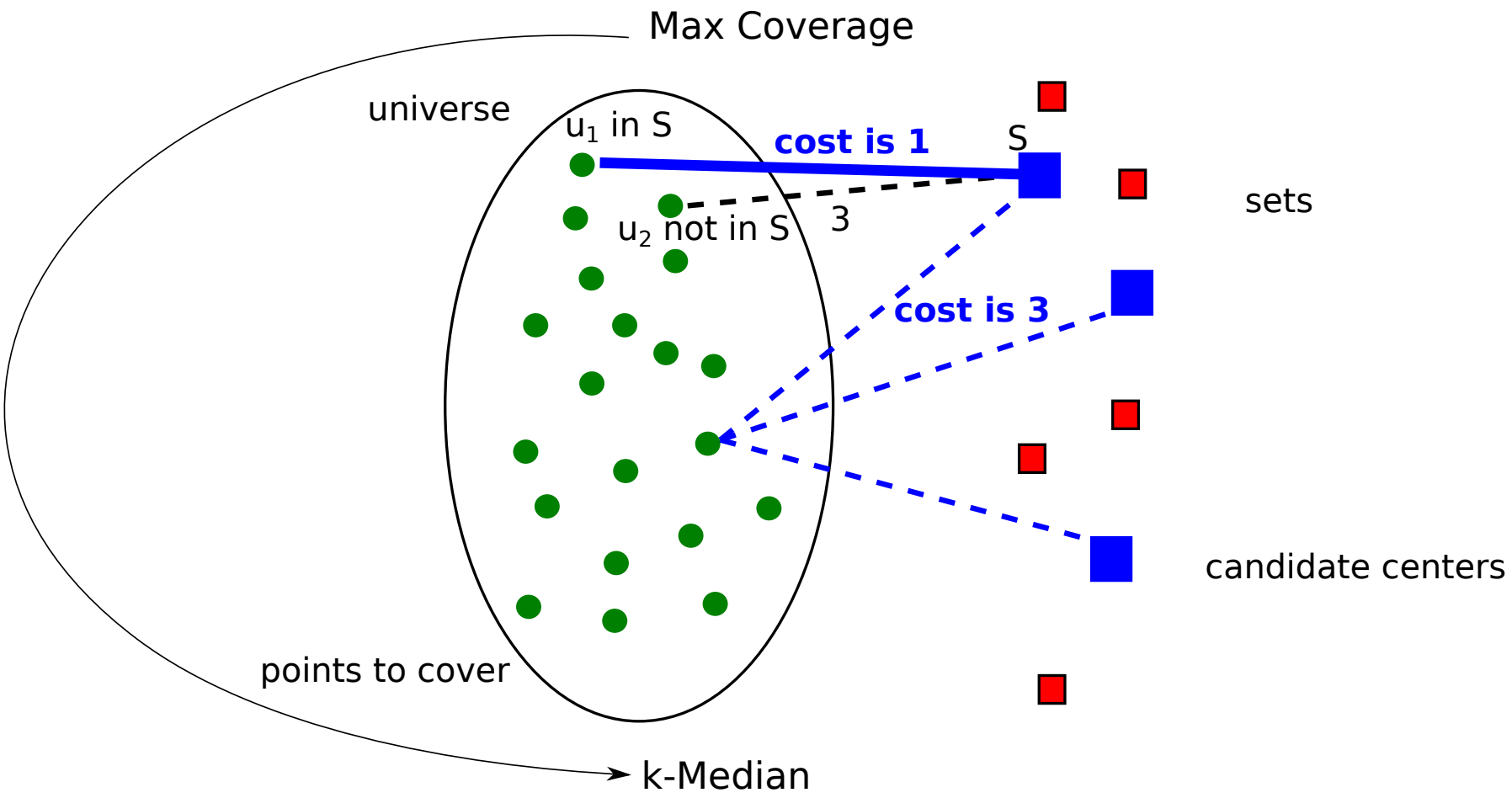

Johnson Coverage Hypothesis

(α, t)-Johnson Coverage Problem

Given $E \subseteq \binom{[n]}{t}$, and k as input, distinguish between:

Completeness: There exists $\mathscr{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ such that

$$\forall T \in E,\ \exists S_i \in \mathscr{C},\ S_i \subset T.$$

Soundness: For every $\mathscr{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ we have

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Johnson Coverage Hypothesis (Cohen-Addad–K–Lee)

$\forall \varepsilon > 0, \exists t_\varepsilon \in \mathbb{N}$ such that $(1 - \frac{1}{e} + \varepsilon, t_\varepsilon)$-Johnson Coverage problem is NP-hard.

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- Recently resolved for $t = 3$
- Improved SDP gaps for Clustering

Embedding in Hamming metric

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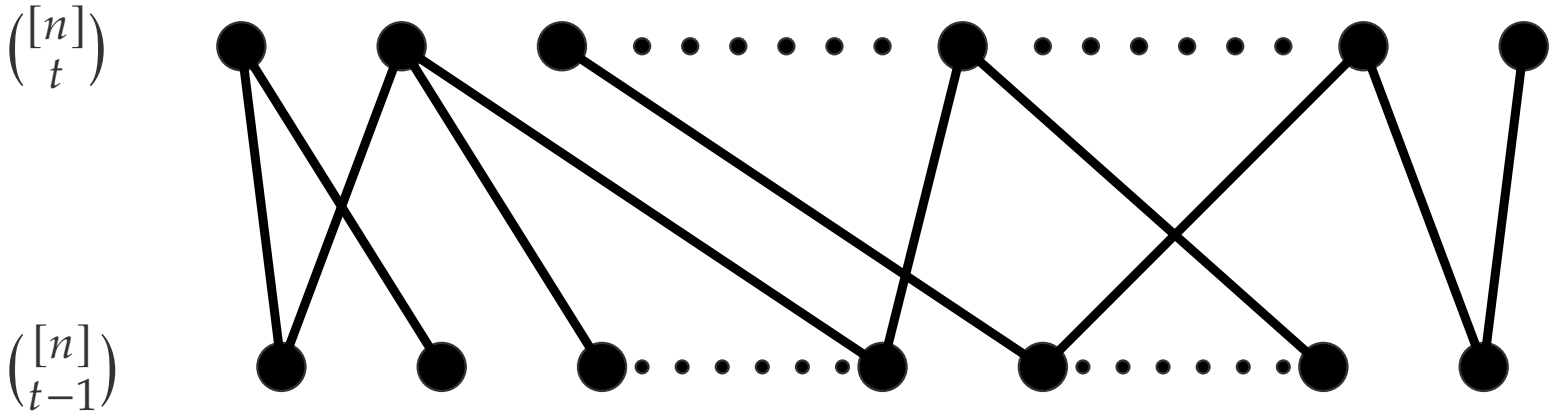

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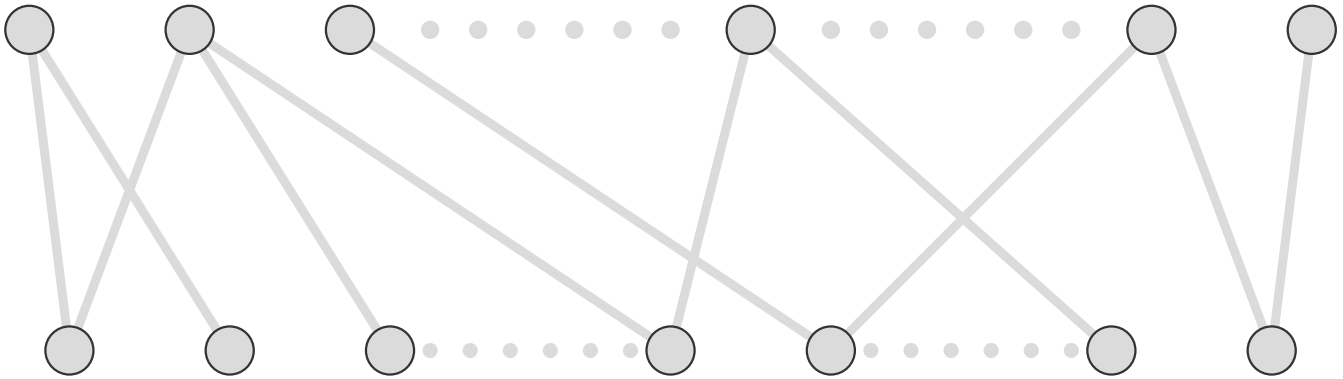

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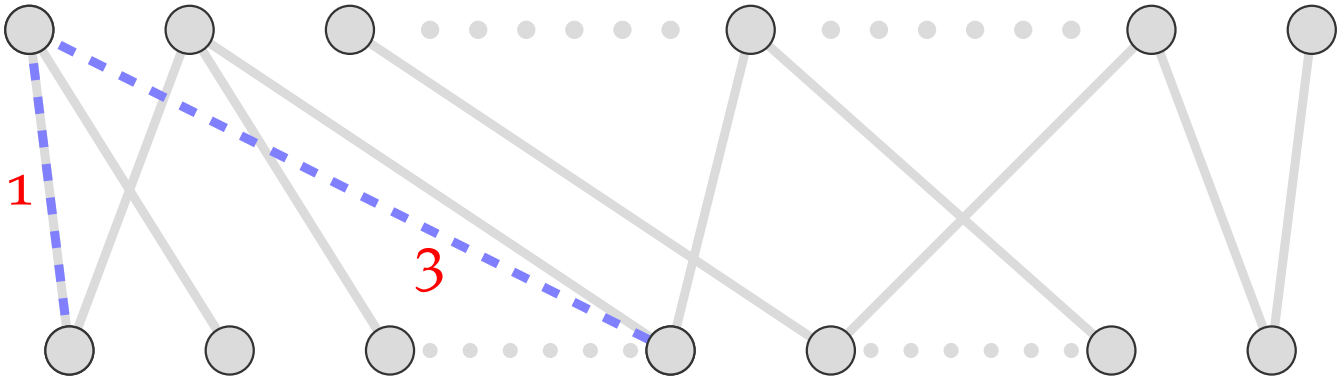

Containment Game

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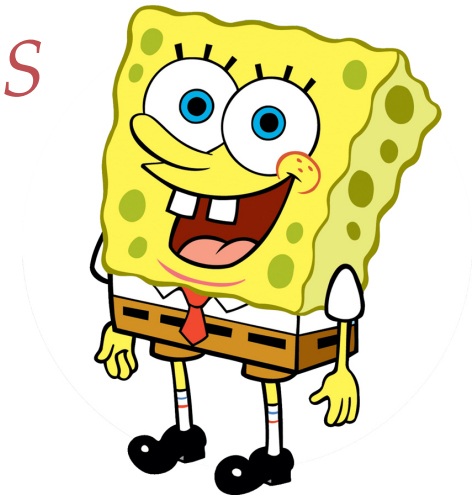

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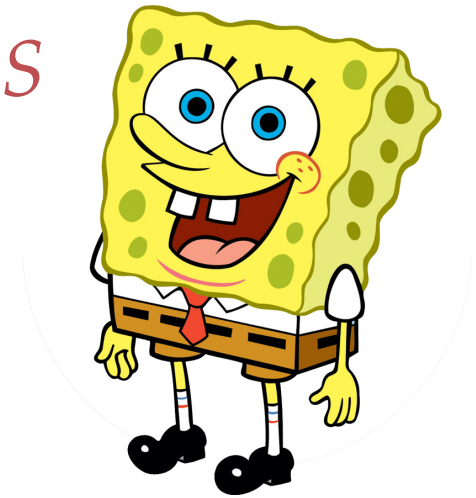

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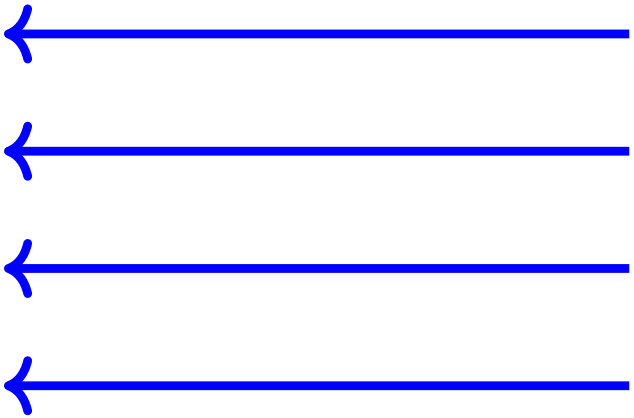
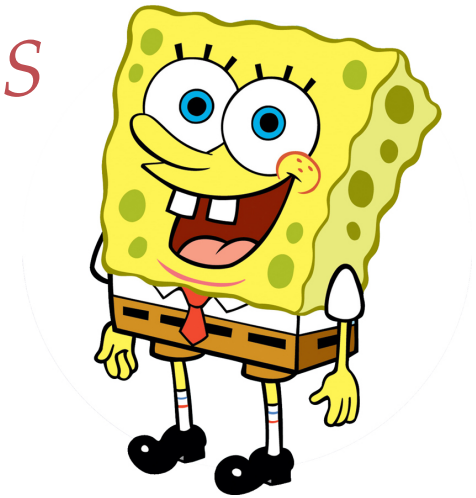


Public Randomness



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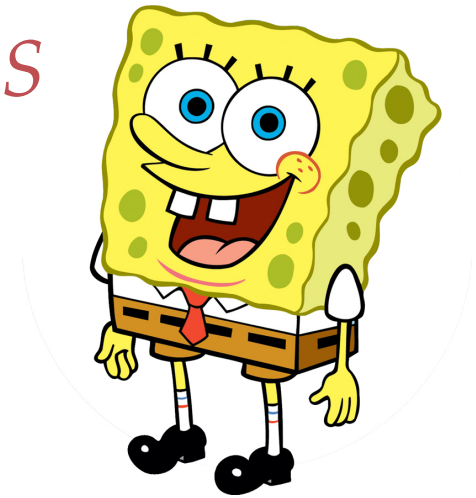

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GOAL

Determine if $S \subset T$

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Embedding Transcript into Hamming metric

- Construct $\tau : 2^{[n]} \to \{0,1\}^{q \cdot c \cdot \log n}$

$$\vdots$$

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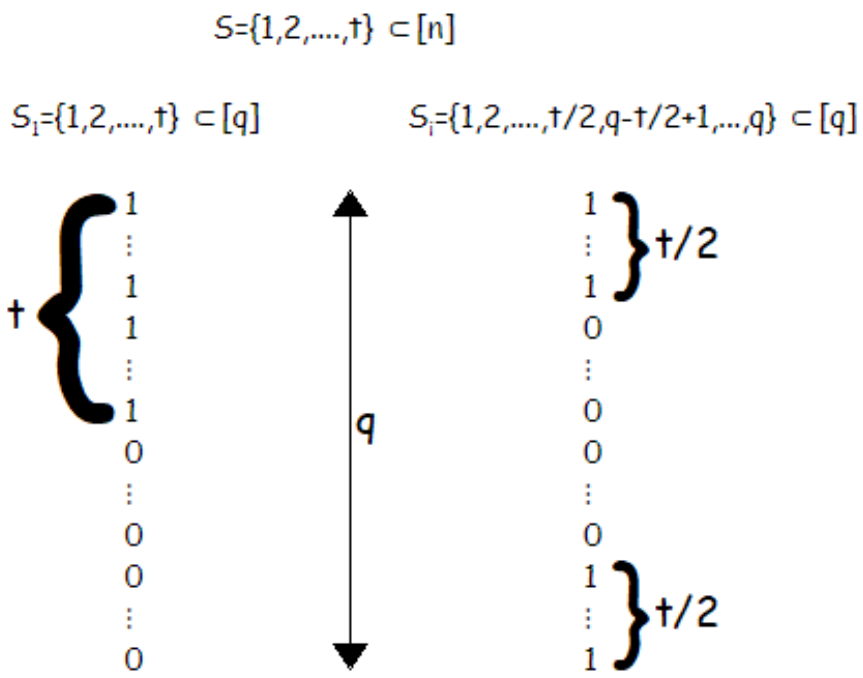

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Structural Observations

Suppose $S \subset T$

For every block i, we have $S_i \subset T_i$

$S_i = \{1,2,\ldots,t/2,q-t/2+1,\ldots,q\} \subset [q]$ $\qquad$ $T_i = S_i \cup \{t+1\} \subset [q]$

1	1
⋮	⋮
1	1
0	0
⋮	⋮
0	0
0	1
0	0
⋮	⋮
0	0
1	1
⋮	⋮
1	1

$$\boxed{|\tau(T\_i)-\tau(S\_i)|=1}$$

Structural Observations

Suppose $S \not\subset T$

For most blocks i, we have $S_i \not\subset T_i$

$S_i \setminus T_i = \{q\}$	$T_i \setminus S_i = \{t+1, t+2\}$
1	1
⋮	⋮
1	1
0	0
⋮	⋮
0	0
0	1
0	1
0	0
⋮	⋮
0	0
1	1
⋮	⋮
1	1
1	0

$$\boxed{|\tau(T\_i)-\tau(S\_i)| \geq 3}$$

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Open: Is JCH true?

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Bonus: Minsum (Definition)

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- <u>Input</u>: $X \subseteq \Gamma$, $k \in \mathbb{N}$

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- Hardness: $1 + \varepsilon$ [Guruswami-Indyk'03]

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Key Ingredient: Hard Instances of Max-Coverage with large girth

Bonus: State-of-the-art Table

Metric	Discrete k-means	Discrete k-median	Continuous k-means	Continuous k-median	k-Minsum
General	3.94 [GK99]	1.73 [GK99]	4	2	1.415
ℓ_0	3.94*	1.73*	2.10*	1.36*	APX-hard [GI03]
ℓ_1	3.94*	1.73*	1.46*	1.36*	APX-hard [GI03]
ℓ_2	1.73*	1.27*	1.36*	1.04*	APX-hard [GI03]
ℓ_∞	3.94	1.73	4	2	1.415

*Under JCH

THANK YOU!