Hardness of Approximation meets Parameterized Complexity

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Global Outline

Day 1: The Setting

Day 2: Gap Creation

Day 3: Applications
Part 1: Hardness of Approximating MaxCover
  - Recap
  - MaxCover with Projection Property
  - Gap Creation
Day 2 Outline

Part 1: Hardness of Approximating MaxCover
- Recap
- MaxCover with Projection Property
- Gap Creation

Part 2: Hardness of Approximating One-Sided Biclique
- Recap
- Gap Creation
Part 1

Gap Creation in MaxCover
MaxCover: Recap

\[
\Gamma(U, W, E) = \left\{ \sum_{i=1}^{r} |U_i| + \sum_{j=1}^{k} |W_j| \right\}
\]

Determine if \( \text{MaxCover}(\Gamma) = 1 \) or \( \text{MaxCover}(\Gamma) \leq s \).

Each \( W_i \) is a Right Super Node.

Each \( U_i \) is a Left Super Node.

\( S \subseteq W \) is a labeling of \( W \) if \( \forall i \in [k], |S \cap W_i| = 1 \).

\( S \) covers \( U_i \) if \( \exists u \in U_i, \forall v \in S, (u, v) \in E \).

\( \text{MaxCover}(\Gamma, S) = \text{Fraction of } U_i \text{'s covered by } S \).

\( \text{MaxCover}(\Gamma) = \max S \text{MaxCover}(\Gamma, S) \).
MaxCover: Recap

Each $W_i$ is a Right Super Node
Each $U_i$ is a Left Super Node

$\Gamma(U, W, E)$
MaxCover: Recap

Each $W_i$ is a \textbf{Right Super Node}

Each $U_i$ is a \textbf{Left Super Node}

$S \subseteq W$ is a \textbf{labeling} of $W$ if

\[ \forall i \in [k], |S \cap W_i| = 1 \]
MaxCover: Recap

Each $W_i$ is a **Right Super Node**
Each $U_i$ is a **Left Super Node**

$S \subseteq W$ is a **labeling** of $W$ if
\[ \forall i \in [k], |S \cap W_i| = 1 \]

$S$ covers $U_i$ if
\[ \exists u \in U_i, \forall v \in S, (u, v) \in E \]
MaxCover: Recap

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MaxCover: Recap

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MaxCover($\Gamma$) = $\max_S$ MaxCover($\Gamma, S$)

Determine if MaxCover($\Gamma$) = 1
or MaxCover($\Gamma$) $\leq s$
MaxCover: Projection Property

\[ \Gamma(U, W, E) \]

\[ \Gamma \] has projection property:

For every \( U_i \) and \( W_j \),

- Induced subgraph of \( (U_i, W_j) \) is:
  - complete bipartite graph (i.e., irrelevant), or,
  - \( \forall w \in W_j, \deg(w) = 1 \) (i.e., projection)
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MaxCover: Projection Property

\[ \Gamma(\mathcal{U}, \mathcal{W}, \mathcal{E}) \]

\( \Gamma \) has projection property:

For every \( U_i \) and \( W_j \),

Induced subgraph of \((U_i, W_j)\) is:

- complete bipartite graph (i.e., irrelevant), or,

- \( \forall w \in W_j, \deg(w) = 1 \) (i.e., projection)
MaxCover with Projection Property is $W[1]$-Hard

Input: $G([n], E_0)$
MaxCover with Projection Property is $W[1]$-Hard

Input: $G([n], E_0)$

$U_i = [n]$ and $W_{j,j'} = E_0$
MaxCover with Projection Property is $W[1]$-Hard

Input: $G([n], E_0)$

$U_i = [n]$ and $W_{j,j'} = E_0$

$W_{j,j'}$ has projection to $U_j$ and $U_{j'}$
Inapproximability of MaxCover \([K-\text{LivniNavon’21}]\)

There is a FPT reduction from MaxCover instance \(\Gamma_0 = (U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0)\) with projection property to a MaxCover instance \(\Gamma = \left(U = \bigcup_{j=1}^{\mathcal{O}(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E\right)\) such that

If \(\text{MaxCover}(\Gamma_0) = 1\) then \(\text{MaxCover}(\Gamma) = 1\)

If \(\text{MaxCover}(\Gamma_0) < 1\) then \(\text{MaxCover}(\Gamma) \leq 0.75|\Gamma|\)

\(\tilde{O}(2^r \cdot |W| \cdot \log |U_0|)\)
Inapproximability of MaxCover \([K\text{-}LivniNavon'21]\)

There is a FPT reduction from MaxCover instance \(\Gamma_0 = \left(U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0\right)\) with projection property to a MaxCover instance \(\Gamma = \left(U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E\right)\) such that

- If MaxCover\((\Gamma_0) = 1\) then MaxCover\((\Gamma) = 1\)
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Inapproximability of MaxCover [K-LivniNavon’21]

There is a FPT reduction from MaxCover instance \( \Gamma_0 = \left( U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0 \right) \) with projection property to a MaxCover instance \( \Gamma = \left( U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E \right) \) such that

- If MaxCover(\( \Gamma_0 \)) = 1 then MaxCover(\( \Gamma \)) = 1
- If MaxCover(\( \Gamma_0 \)) < 1 then MaxCover(\( \Gamma \)) \leq 0.75
- |\( \Gamma \)| = \( \tilde{O}(2^r \cdot |W| \cdot \log |U_0|) \)
- The reduction runs in time \( 2^{O(r)} \cdot \text{poly}(|\Gamma_0|) \).
Coding Theory: Recap

- \( C \subseteq [q]^L \)

- **Distance** of \( C \):
  \[
  \Delta(C) := \min_{x,y \in C} \| x - y \|_0
  \]
Coding Theory: Recap

- $C \subseteq [q]^L$

- **Distance** of $C$:

\[ \Delta(C) := \min_{x, y \in C} \|x - y\|_0 \]

- For some constant $\rho > 0$, collection of $2^{\rho L}$ Random Binary Strings is a code with distance $L/4$
Coding Theory: Recap

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- **Distance** of $C$:
  $$\Delta(C) := \min_{x, y \in C} \|x - y\|_0$$

- For some constant $\rho > 0$, collection of $2^{\rho L}$ Random Binary Strings is a code with distance $L/4$

- **Reed Solomon Codes**:
  - Evaluations of degree $d$ univariate polynomials over $\mathbb{F}_q$
  - $|RS| = q^{d+1}$
  - $\Delta(RS) = q - d$
  - $q^{d+1}$ codewords in $[q]^q$ with distance $q - d$
Threshold Graph Construction

\[ A_t = \{0, 1\}^r \]

\[ U_i^0 = C \]
Threshold Graph Construction

$\mathcal{A} = \{0, 1\}^r$

$U_0^0 = C$

$W_1$

$W_2$

$W_k$

$(u, (q_1, \ldots, q_r)) \in U_i^0 \times A_t$ is an edge $\iff$ $u_t = q_i$
Threshold Graph Properties

Completeness

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\) there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\)
Threshold Graph Properties

Completeness

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\) there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\).

Soundness

For every \(u, u' \in U_i^0\), there are at most \(L - \Delta(C)\) many supernodes in \(A\) which have a common neighbor of \(u\) and \(u'\).
Threshold Graph Composition

$A_t = \{0, 1\}^r$

$U_i^0 = C$

Parameterized Inapproximability

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Threshold Graph Composition

\[ A_t = \{0, 1\}^r \]

\((w, (q_1, \ldots, q_r)) \in W_j \times A_t\) is an edge if and only if there exists \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) such that for all \(i \in [k]\), \((w, u^i)\) and \((u^i, (q_1, \ldots, q_r))\) are both edges.
Let \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\) be optimal labeling of \(\Gamma_0\).

Let \((u^1, \ldots, u^r) \in U^0_1 \times \cdots \times U^0_r\) be common neighbors of \((w_1, \ldots, w_k)\) in \(\Gamma_0\).
Completeness of Reduction

- Let \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\) be optimal labeling of \(\Gamma_0\)

- Let \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) be common neighbors of \((w_1, \ldots, w_k)\) in \(\Gamma_0\)

Completeness of Threshold Graph

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\) there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\)
Soundness of Reduction

- Fix \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\)
- There exists \(U_i^0\) not covered by \((w_1, \ldots, w_k)\)
Soundness of Reduction

- Fix \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\)
- There exists \(U_i^0\) not covered by \((w_1, \ldots, w_k)\)
- There exists \(w_j\) and \(w_j'\) with neighbors \(u\) and \(u'\) resp. in \(U_i^0\) \((u \neq u')\)
Soundness of Reduction

- Fix \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\)
- There exists \(U_i^0\) not covered by \((w_1, \ldots, w_k)\)
- There exists \(w_j\) and \(w'_j\) with neighbors \(u\) and \(u'\) resp. in \(U_i^0\) \((u \neq u')\)
- If \(a \in A\) is common neighbor of \(w_j\) and \(w'_j\) in \(\Gamma\) then \(u\) and \(u'\) are common neighbors of \(a\) in Threshold graph.
Soundness of Reduction

- Fix \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\)
- There exists \(U_i^0\) not covered by \((w_1, \ldots, w_k)\)
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- If \(a \in A\) is common neighbor of \(w_j\) and \(w_j'\) in \(\Gamma\) then \(u\) and \(u'\) are common neighbors of \(a\) in Threshold graph

Soundness of Threshold Graph

For every \(u, u' \in U_i^0\), there are at most \(L - \Delta(C)\) many supernodes in \(A\) which have a common neighbor of \(u\) and \(u'\)
MaxCover: Gap Creation

Inapproximability of MaxCover using Random Binary Codes

There is a FPT reduction from MaxCover instance $\Gamma_0 = \left( U_0 = \bigcup_{j=1}^{r} U^0_j, W = \bigcup_{j=1}^{k} W_i, E_0 \right)$ with projection property to a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$ such that

- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
- If $\text{MaxCover}(\Gamma_0) < 1$ then $\text{MaxCover}(\Gamma) \leq 0.75$
- $|\Gamma| = \tilde{O}(2^r \cdot |W| \cdot \log |U_0|)$
- The reduction runs in time $2^{O(r)} \cdot \text{poly}(|\Gamma_0|)$. 
Threshold Graph Composition with Reed Solomon Codes

\[ A_t = [q]^r \]

\[(w, (q_1, \ldots, q_r)) \in W_j \times A_t \text{ is an edge } \iff \exists (u^1, \ldots, u^r) \in U^0_1 \times \cdots U^0_r \text{ such that } \forall i \in [k], (w, u^i) \text{ and } (u^i, (q_1, \ldots, q_r)) \text{ are both edges} \]
Threshold Graph Properties

Completeness

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\) there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\)

Soundness

For every \(u, u' \in U_i^0\), there are at most \(\log_q |U_0|\) many supernodes in \(A\) which have a common neighbor of \(u\) and \(u'\)
Inapproximability of MaxCover using Reed Solomon Codes

There is a FPT reduction from MaxCover instance \( \Gamma_0 = \left( U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0 \right) \) with projection property to a MaxCover instance \( \Gamma = \left( U = \bigcup_{j=1}^{q} U_j, W = \bigcup_{j=1}^{k} W_i, E \right) \) such that:

- If \( \text{MaxCover}(\Gamma_0) = 1 \) then \( \text{MaxCover}(\Gamma) = 1 \)
- If \( \text{MaxCover}(\Gamma_0) < 1 \) then \( \text{MaxCover}(\Gamma) \leq \frac{\log_q |U_0|}{q} \)
- \( |\Gamma| = \tilde{O}(q^r \cdot |W| \cdot \log |U_0|) \)
- The reduction runs in time \( q^r \cdot \text{poly}(|\Gamma_0|) \).
Part 2

Gap Creation in One-Sided Biclique
One-Sided Biclique: Recap

\[ \Gamma(U, W, E) \]
One-Sided Biclique: Recap

\[ \Gamma(U, W, E) \]

Find \( k \) vertices in \( W \) with most common neighbors
One-Sided Biclique: Recap

Find $k$ vertices in $W$ with most common neighbors
Inapproximability of One-Sided Biclique (Lin’18)

There is a FPT reduction from $k$-Clique instance $G([n], E_0)$ to a One-Sided Biclique instance $\Gamma = (U, W, E)$ such that:

- If $G$ has a $k$-clique then there are $\binom{k}{2}$ vertices in $W$ which have $n^{1/k}$ common neighbors in $U$.
- If $G$ has no $k$-clique then for every $\binom{k}{2}$ vertices in $W$ they have at most $(k + 1)!$ common neighbors in $U$.
- $|\Gamma| = n^3$.
- The reduction runs in time $\text{poly}(n)$.
Starting from $k$-Clique

\[ V = \{\{n\}\} \quad W = E_0 \]

Input: $G([n], E_0)$

- If $G$ has a $k$-clique then there are $(\binom{k}{2})$ vertices in $W$ which in total have $k$ neighbors.
- If $G$ has no $k$-clique then any $(\binom{k}{2})$ vertices in $W$ has totally at least $k+1$ neighbors.
Starting from $k$-Clique

$V = [n]$

$W = E_0$

Input: $G([n], E_0)$

If $G$ has a $k$-clique then there are $\binom{k}{2}$ vertices in $W$ which in total have $k$ neighbors.
Starting from $k$-Clique

Input: $G([n], E_0)$

If $G$ has a $k$-clique then there are \( \binom{k}{2} \) vertices in $W$ which in total have $k$ neighbors.

If $G$ has no $k$-clique then any \( \binom{k}{2} \) vertices in $W$ has totally at least $k+1$ neighbors.
Threshold Graph

\[ A = [n] \]
\[ V = [n] \]

Every \( k \) vertices in \( V \) has at least \( \frac{n}{k} \) common neighbors in \( A \).

Every \( k+1 \) vertices in \( V \) has at most \( (k+1)! \) common neighbors in \( A \).
A threshold graph is defined by:

- \( A = [n] \)
- \( V = [n] \)

Every \( k \) vertices in \( V \) has at least \( n^{1/k} \) common neighbors in \( A \).
Threshold Graph

$A = [n]$

$V = [n]$

Every $k$ vertices in $V$ has at least $n^{1/k}$ common neighbors in $A$

Every $k+1$ vertices in $V$ has at most $(k + 1)!$ common neighbors in $A$
Threshold Graph Composition

\[ A = \{1, \ldots, n\} \]

\[ V = \{1, \ldots, n\} \]

\[ W = E_0 \]

A graph is an edge \( \in W \times A \) if and only if there exist \( v, v' \in V \) such that \( a \) and \( w \) are common neighbors of \( v \) and \( v' \).
\( W = E_0 \)
\( A = [n] \)
\( V = [n] \)

\((w, a) \in W \times A \) is an edge \( \iff \exists v, v' \in V \) such that 
a and \( w \) are common neighbors of \( v \) and \( v' \)
Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of $k$-clique in $G$

- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph
Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of $k$-clique in $G$
- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph
- Every $a \in A'$ is also a common neighbor of $e_{v_i,v_j} \in W$ in $\Gamma$
Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of $k$-clique in $G$
- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph
- Every $a \in A'$ is also a common neighbor of $e_{v_i,v_j} \in W$ in $\Gamma$

Completeness of Threshold Graph

Every $k$ vertices in $V$ has at least $n^{1/k}$ common neighbors in $A$
Soundness of Reduction

- Fix \((w_1, \ldots, w_{\binom{k}{2}}) \in W\) and let \(A' \subseteq A\) be its set of common neighbors in \(\Gamma\)
Soundness of Reduction

- Fix \((w_1, \ldots, w_{\binom{k}{2}}) \in W\) and let \(A' \subseteq A\) be its set of common neighbors in \(\Gamma\).
- Let \(V' \subseteq V\) be set of total neighbors of \((w_1, \ldots, w_{\binom{k}{2}})\) in \(V\).
- \(|V'| \geq k + 1\)
Soundness of Reduction

- Fix \((w_1, \ldots, w_{\binom{k}{2}}) \in W\) and let \(A' \subseteq A\) be its set of common neighbors in \(\Gamma\).
- Let \(V' \subseteq V\) be set of total neighbors of \((w_1, \ldots, w_{\binom{k}{2}})\) in \(V\).
- \(|V'| \geq k + 1\)
- \(A'\) is a subset of the common neighbors of \(V'\) in Threshold graph.
Soundness of Reduction

- Fix \((w_1, \ldots, w_{\binom{k}{2}}) \in W\) and let \(A' \subseteq A\) be its set of common neighbors in \(\Gamma\).
- Let \(V' \subseteq V\) be set of total neighbors of \((w_1, \ldots, w_{\binom{k}{2}})\) in \(V\).
- \(|V'| \geq k + 1\)
- \(A'\) is a subset of the common neighbors of \(V'\) in Threshold graph.

Soundness of Threshold Graph

Every \(k+1\) vertices in \(V\) has at most 
\((k + 1)!\) common neighbors in \(A\).
Inapproximability of One-Sided Biclique (Lin’18)

There is a FPT reduction from $k$-Clique instance $G([n], E_0)$ to a One-Sided Biclique instance $\Gamma = (U, W, E)$ such that

- If $G$ has a $k$-clique then there are $\binom{k}{2}$ vertices in $W$ which have $n^{1/k}$ common neighbors in $U$.
- If $G$ has no $k$-clique then for every $\binom{k}{2}$ vertices in $W$ they have at most $(k + 1)!$ common neighbors in $U$.
- $|\Gamma| = n^3$.
- The reduction runs in time $\text{poly}(n)$. 
Threshold Graph

\[ A = [n] \]

\[ V = [n] \]

Every \( k \) vertices in \( V \) has at least \( n^{1/k} \) common neighbors in \( A \)

Every \( k+1 \) vertices in \( V \) has at most \((k + 1)!\) common neighbors in \( A \)
Threshold Graph

$A = [n]$

$V = [n]$

Every $k$ vertices in $V$ has at least $n/\binom{k}{2}$ common neighbors in $A$

Every $k + 1$ vertices in $V$ has at most $(k + 1)!$ common neighbors in $A$

IT DOES EXIST
Random Algebraic Constructions

- Erdős-Renyi model Random graphs fail: long smooth-decaying tail
Random Algebraic Constructions

- Erdös-Renyi model Random graphs fail: long smooth-decaying tail
- Random graphs defined over some specific ‘algebraic distribution’ suffice
Random Algebraic Constructions

- Erdős-Renyi model Random graphs fail: long smooth-decaying tail
- Random graphs defined over some specific ‘algebraic distribution’ suffice
- Normed graphs provide semi-explicit construction
Take-away Intuition and Remarks

- Threshold Graph Composition Technique Ingredients:
  - Threshold Graph
  - Composition of Input Graph with Threshold Graph
Take-away Intuition and Remarks

- Threshold Graph Composition Technique Ingredients:
  - Threshold Graph
  - Composition of Input Graph with Threshold Graph
- Threshold Graph
  - What are the required threshold properties?
  - Does the graph with above properties exist?
Take-away Intuition and Remarks

Threshold Graph Composition Technique Ingredients:

- Threshold Graph
- Composition of Input Graph with Threshold Graph

Threshold Graph

- What are the required threshold properties?
- Does the graph with above properties exist?

Tweak ‘Composition of Input Graph with Threshold Graph’ in order to require weaker/more realistic threshold properties
Take-away Intuition and Remarks

- Threshold Graph Composition Technique Ingredients:
  - Threshold Graph
  - Composition of Input Graph with Threshold Graph

- Threshold Graph
  - What are the required threshold properties?
  - Does the graph with above properties exist?

- Tweak ‘Composition of Input Graph with Threshold Graph’ in order to require weaker/more realistic threshold properties

- Start from more structured Input problem
Tomorrow's plan

- Set Cover
- Biclique
- Clique