Hardness of Approximation meets Parameterized Complexity

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Global Outline

Day 1: The Setting

Day 2: Gap Creation

Day 3: Applications

Day 1 Outline

Part 1: Hardness of Approximation

- Hardness of Approximation in NP
- Hardness of Approximation in Parameterized Complexity

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Part 2: Key Problems in Parameterized Inapproximability

- MaxCover
- One-Sided Biclique

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- One-Sided Biclique

Part 3: Coding Theory

- Definition and Geometric Intuition
- Random Codes
- Algebraic Codes

Part 1

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PCP Theorem & Label Cover

PCP Theorem: Bedrock of
NP-Hardness of Approximation

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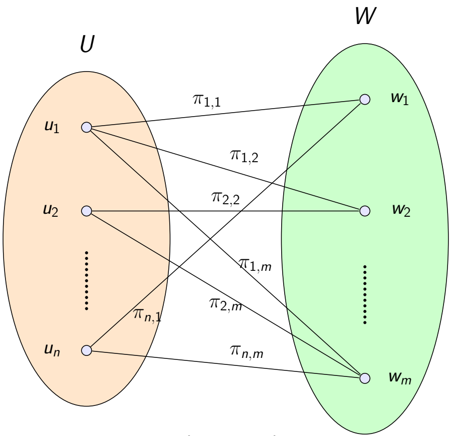

$\Gamma(U, W, E)$

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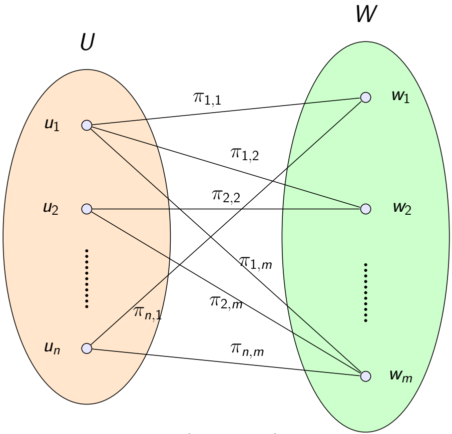
$$\pi_{i,j} \subseteq \Sigma_U \times \Sigma_W$$



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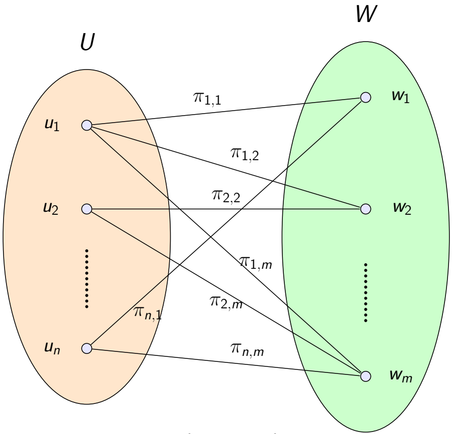

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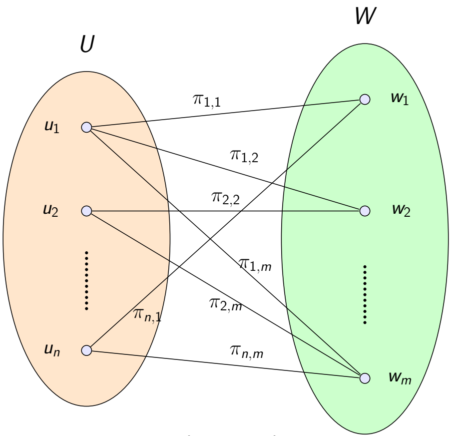

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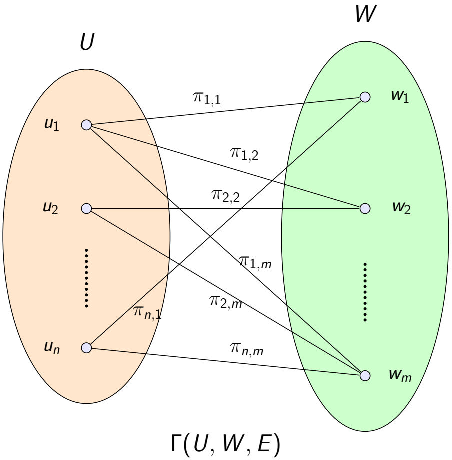

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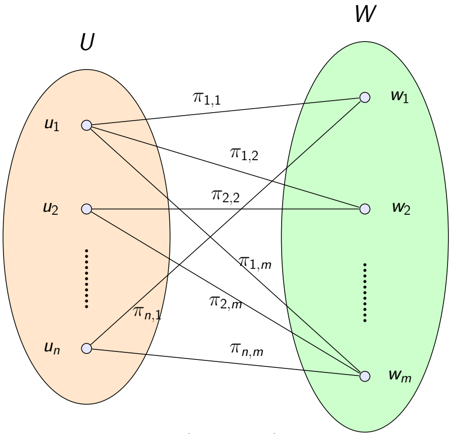

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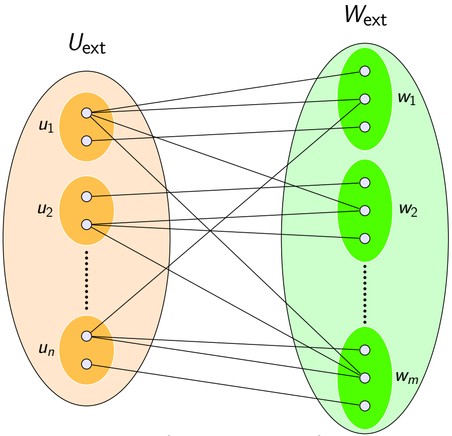
Determining if $\text{VAL}(\Gamma) = 1$ or if $\text{VAL}(\Gamma) \leq 0.99$ is NP-Hard

Extended Label Cover



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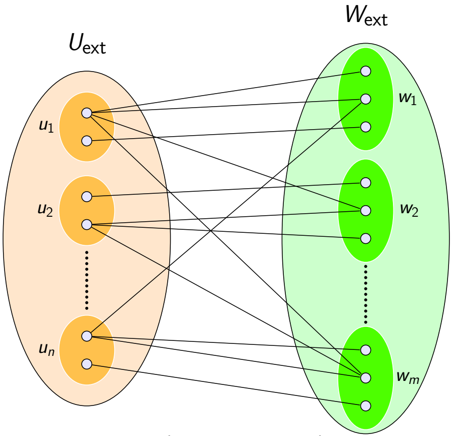

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$n \cdot |\Sigma_U|$ nodes in U
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iff $(u_i, w_j) \in E$ and $(\alpha, \beta) \in \pi_{i,j}$

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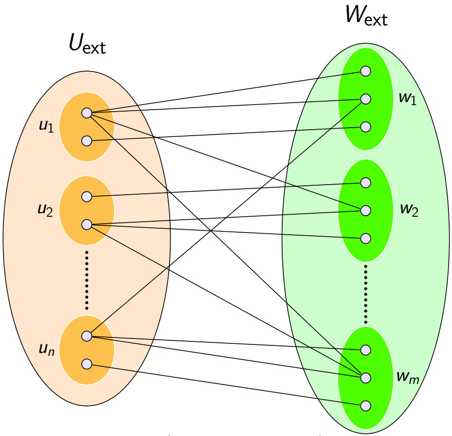

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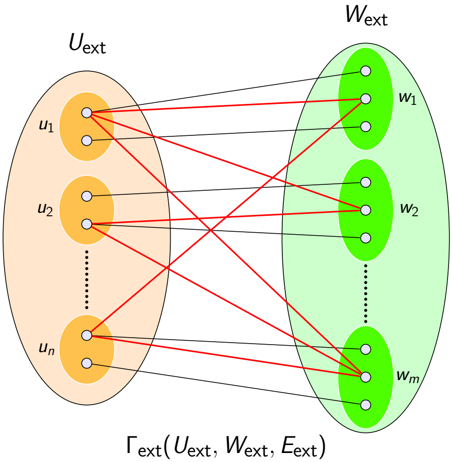

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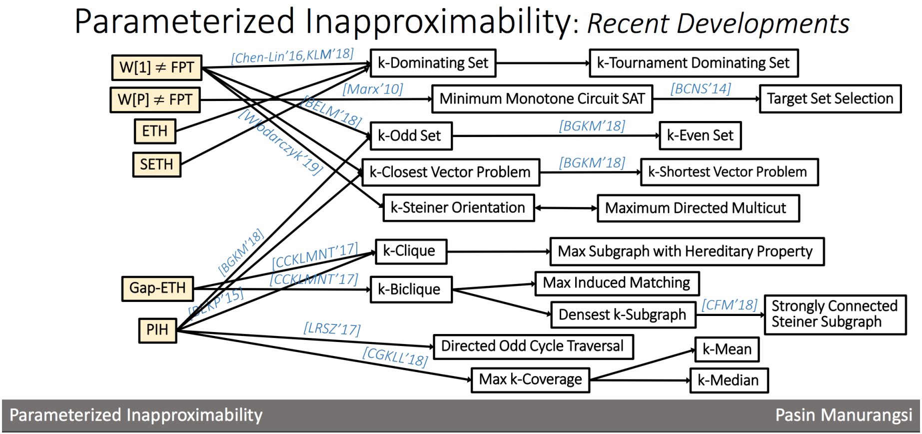
- Set Cover: Hard to cope!
- New direction: Fixed Parameter Approximability

Is there a $F(k) \cdot \text{poly}(n)$ time algorithm
that approximates to a factor $T(k)$?

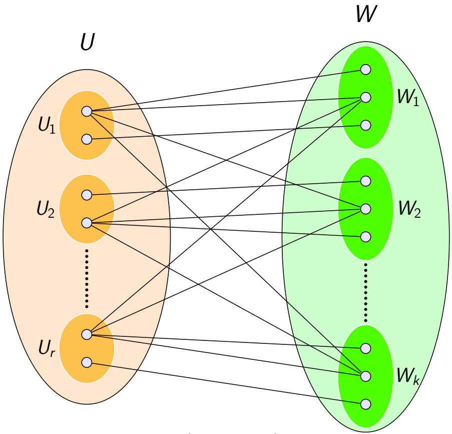
Parameterized Inapproximability: Partial Summary



Part 2

Key Problems in Parameterized Inapproximability

MaxCover [Chalermsook et al. 2017]



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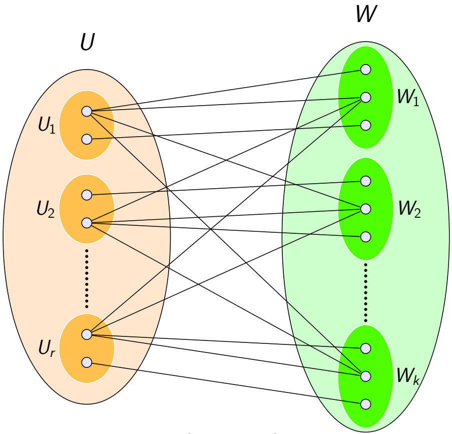

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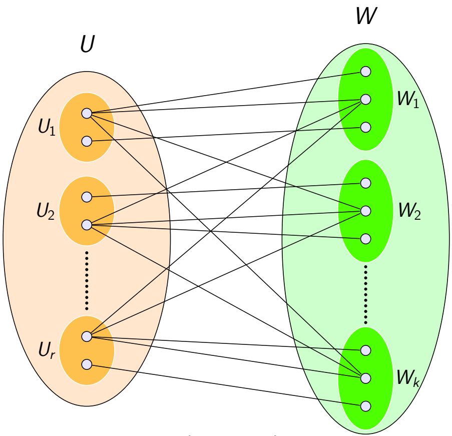

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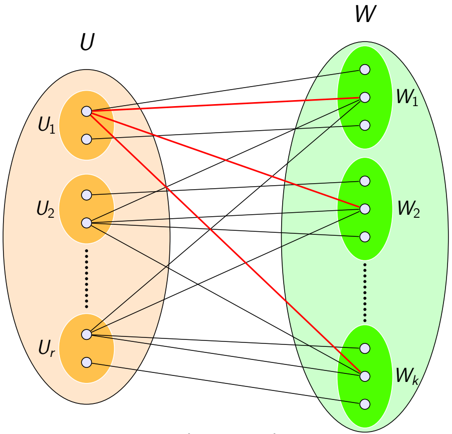

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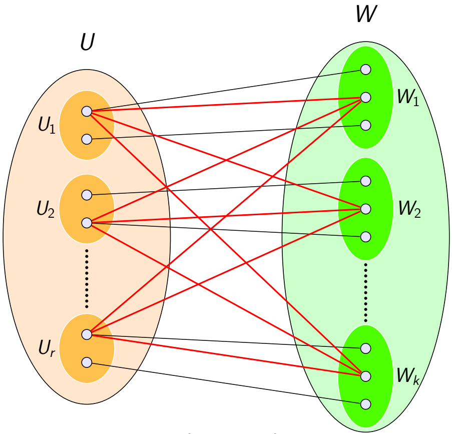

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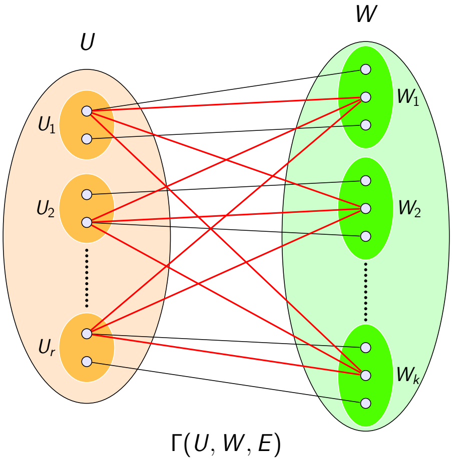

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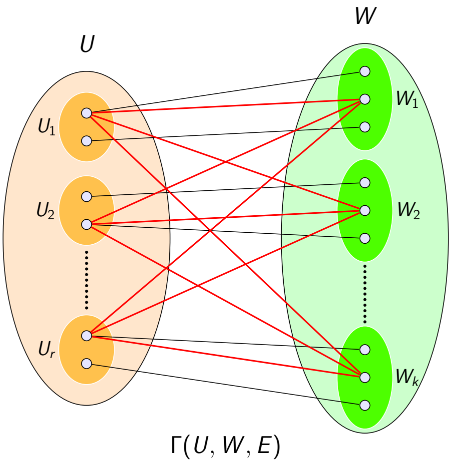

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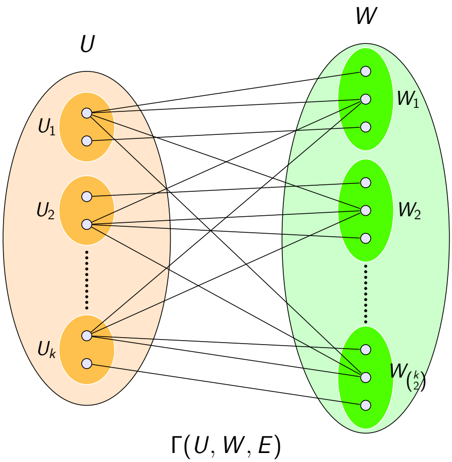
k-Clique as MaxCover



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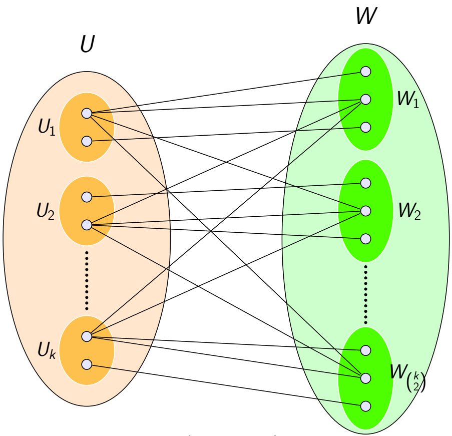

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Input of k-Clique problem:
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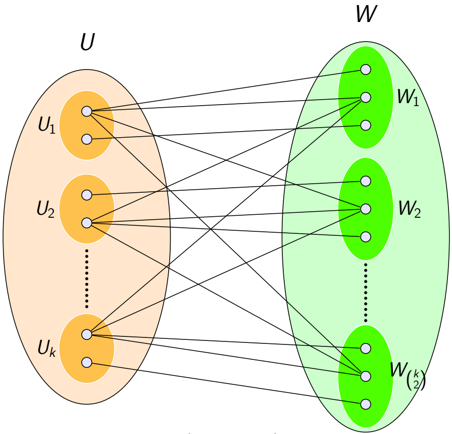

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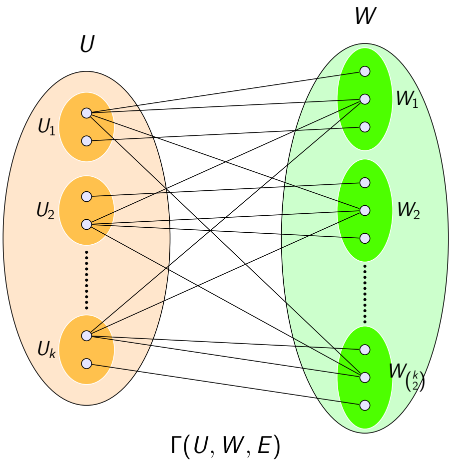

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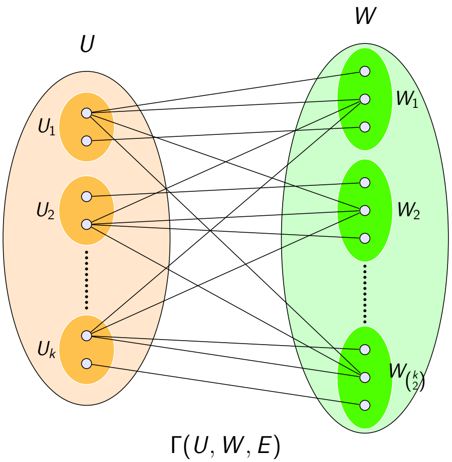

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MaxCover: Results

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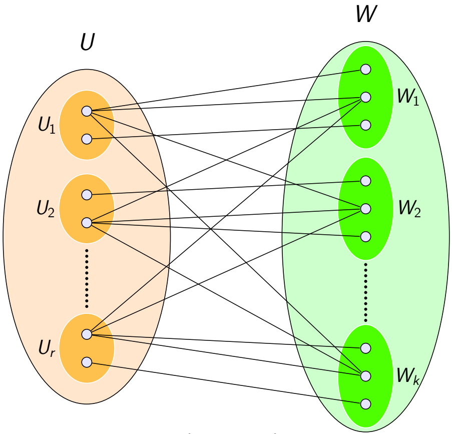
- Central problem to understand parameterized inapproximability of Set Cover and Clique

MaxCover: W[1] Membership



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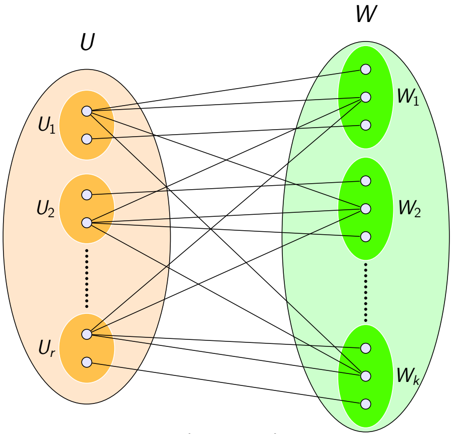

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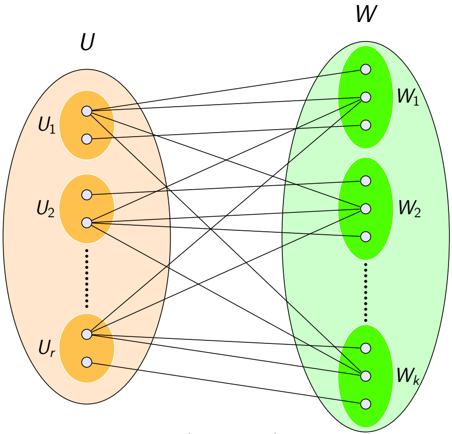

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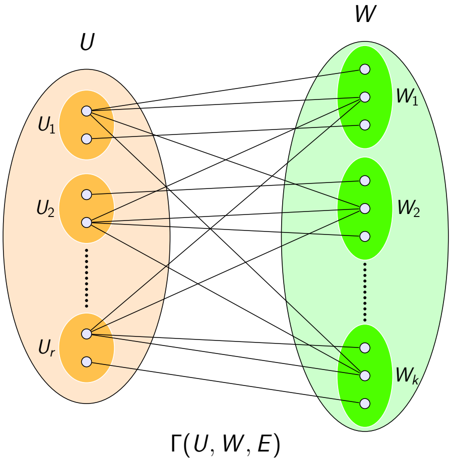

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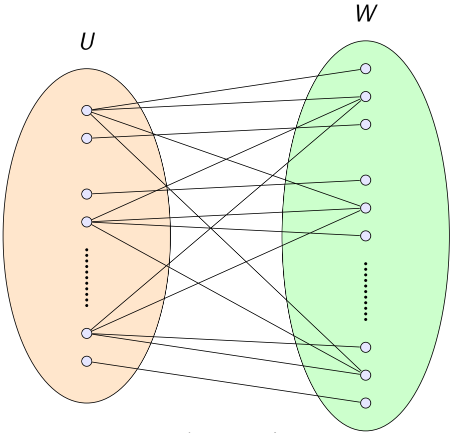
MaxCover from ETH and SETH have $r = F(k)$

One-Sided Biclique



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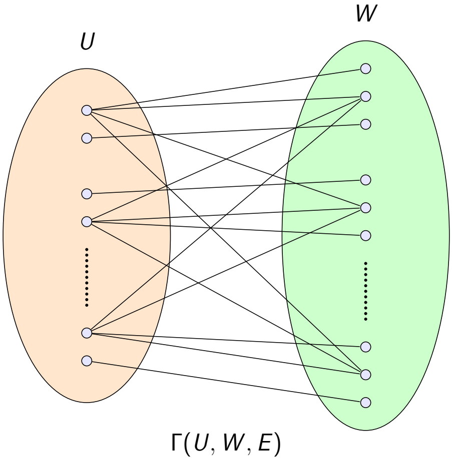

Find k vertices in W
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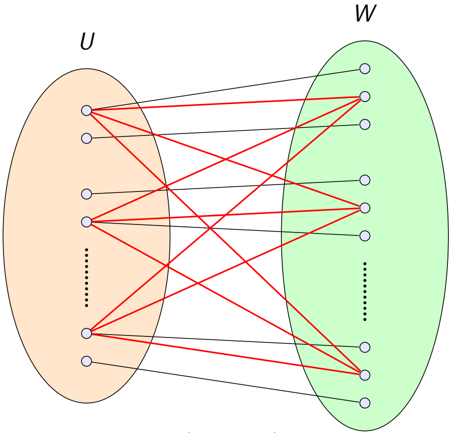

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One-Sided Biclique vs. MaxCover

- Colored vs. Non-colored

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One-Sided Biclique vs. MaxCover

- Colored vs. Non-colored
- Covering vs. Common neighbors
- One-Sided Biclique reduces to MaxCover: Color Coding
    - What about the other direction?

Summary

- Hardness of Approximation meets Parameterized Complexity: New Exciting Area!
- MaxCover and One-Sided Biclique are key problems for which we have proved inapproximaiblity results.

Part 3

Coding Theory

Coding Theory: Geometric Motivation

- Consider all strings/points in $\{0,1\}^n$

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A good code: for $\rho, \delta > 0$, $|C| = 2^{\rho L}$, $\Delta(C) = \delta L$.

Random Codes

Random Strings are Good Codes

For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently then they form a code with distance at least $1/4$ (whp).

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- Chernoff: $\Pr[\|x - y\|_0 \leq L/4] = e^{-L/100}$
- Union Bound:

$$\Pr[\min_{x,y \in C}\{\|x - y\|_0\} \leq L/4] = 2^{2\rho L} e^{-L/100} < 0.001$$

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For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently then they form a code with distance at least $1/4$ (whp).

- $\mathbb{E}[\|x - y\|_0] = L/2$
- Chernoff: $\Pr[\|x - y\|_0 \leq L/4] = e^{-L/100}$
- Union Bound:

$$\Pr[\min_{x,y \in C}\{\|x - y\|_0\} \leq L/4] = 2^{2\rho L} e^{-L/100} < 0.001$$

Many Efficient Deterministic Good Codes Exist!

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- Reed Solomon Codes meet the Singleton bound!

Tomorrow's plan

- MaxCover: Gap Creation by using Codes
- One-Sided Biclique: Gap creation by using Random Graphs/Polynomials